A Zero-Sum Electromagnetic Evader-Interrogator Differential Game with Uncertainty

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In Memoriam of Prof. L. D. Berkovitz

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Abstract

We consider dynamic electromagnetic evasion-interrogation games in which the evader can use ferroelectric material coatings to attempt to avoid detection while the interrogator can manipulate the interrogating frequencies to enhance detection. The resulting problem is formulated as a two-player zero-sum dynamic differential game in which the cost functional is based on the expected value of the intensity of the reflected signal. We show that there exists a saddle point for the relaxed form of this dynamic differential game in which the relaxed controls appear bilinearly in the dynamics governed by a partial differential equation. We also present a computational framework for construction of approximate saddle point strategies in feedback form for a special case of this relaxed differential game with strategies and payoff in the sense of Berkovitz.

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1 Introduction

In an electromagnetic evasion-interrogation game, the evader wishes to minimize the intensity of the reflected signal to remain undetected in carrying out his mission while the interrogator wishes to maximize the intensity of the reflected signal to detect the attacker. The results in [5] demonstrated that it is possible to design ferroelectric/ferromagnetic materials with appropriate dielectric permittivity and magnetic permeability to significantly attenuate reflections of known electromagnetic interrogation signals from highly conductive targets such as airfoils and missiles. However the results in [6] showed that if the evader employed a counter interrogation design based on a fixed set of known interrogating frequencies, then by a rather simple counter-counter interrogation strategy (use of an interrogating frequency little more than 10% different from the assumed evader design frequencies), the interrogator can easily defeat the evader's material coatings counter interrogation strategy to obtain strong reflected signals. Thus, one can readily conclude from these two results that the evader and the interrogator must each try to confuse the other by introducing significant uncertainty in their design and interrogating strategies, respectively.

Based on this consideration, a static electromagnetic evasion-interrogation game (in the spirit of mixed strategies introduced by von Neumann [37]) was considered in [2], where the problem is mathematically formulated as a minimax game over sets of probability measures taken with the Prohorov metric. In this case this is equivalent to the weak star topology for the set of probability measures considered as a subset of the dual C^* of C, the bounded continuous functions with the supremum norm. In this formulation, the evader does not choose a single coating, but rather has a set of possibilities available for choice and only chooses the probabilities with which he will employ the materials on a target. By choosing his coatings randomly (according to a best strategy to be determined in a minimax game), he prevents adversaries from discovering which coating he will use – indeed, even he does not know which coating will be chosen for a given target. The interrogator, in a similar approach, determines best probabilities for choices of frequency and angle in the interrogating signals. Using compactness and approximation properties in the context of the Prohorov metric, the authors in [2] present a rather complete theoretical and computational framework for these static problems. A more realistic (for some scenarios) dynamic setting is initially introduced in [3] by consideration of time dynamics in the problem, wherein the evader is allowed to make dynamic changes to his strategies in response to the dynamic input information with uncertainty on the interrogator's actions.

In this paper, we consider a two player zero-sum differential game in an infinite-dimensional space, where the cost functional is based on the intensity of reflected signals. In this formulation, both evader and interrogator choose a probability measure at each time t in the presence of material uncertainty which is modeled as a stochastic process. The outline of this paper is as follows. In Section 2 we present a description of our problem formulation and show that there exists a saddle point for the resulting relaxed differential game. Then in Section 3 we present a computational framework for construction of approximate saddle-

point strategies in feedback form for a special case of this differential game. Some summary remarks are given in Section 4.

2 Problem Formulation and Saddle Points for the Relaxed Differential Game

The cost functional is based on the intensity of reflected signals from an object such as an airfoil or missile coated by a radar absorbent material of constant thickness. There are several ways to treat the electromagnetic scattering [5, 6]. One fundamental approach is to employ the far field pattern for reflected waves computed directly using Maxwell's equations. As detailed in [2], in two dimensions for a reflecting body with a given coating layer with an interrogating plane wave $E^{(i)}$, the scattered field $E^{(s)}$ satisfies the Helmholtz equation [12]. An alternative and much less computationally expensive one (as well as equally accurate in this setting – see [5, 6]) is to calculate the reflection coefficient based on a simple planar geometry (e.g., see Fig. 1) with Fresnel's formula for a perfectly conducting half plane.

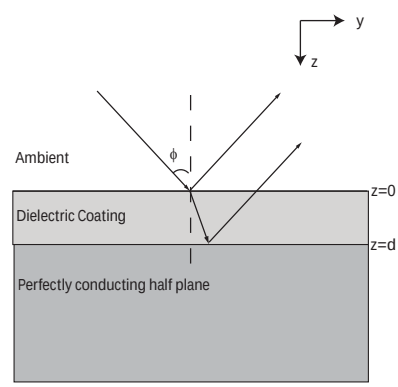

Figure 1: Interrogating high frequency wave impinging (angle of incidence ϕ) on coated (thickness d) perfectly conducting surface

We will use the reflection coefficient to measure the strength of backscattering. We assume that a normally incident electromagnetic wave with the angular frequency ω is assumed to impinge the half plane. Then the corresponding wave length in the air is $2\pi c/\omega$, where the speed of light is $c = 3 \times 10^8$. Thus, the reflection coefficient R for a wave impinging on a coating layer of thickness d with relative dielectric permittivity ϵ and relative magnetic permeability μ is given by

$$R(\mu, \epsilon, \omega, d) = \frac{r_1 + r_2}{1 + r_1 r_2}, \tag{2.1}$$

where

$$r_1 = \frac{\epsilon - \sqrt{\epsilon\mu}}{\epsilon + \sqrt{\epsilon\mu}} \qquad \text{and} \qquad r_2 = \exp\left(2i\sqrt{\epsilon\mu}\omega d/c\right). \tag{2.2}$$

This expression can be derived directly from Maxwell's equation by considering the ratio of reflected to incident waves, for example, in the case of parallel polarized (TE_x) incident wave (e.g., see [5, 23]).

Control of reflections by the evader is effected via local currents in a composite layered reflector device that can be used to control the dielectric permittivity and magnetic permeability in a target coating layer as discussed above and in more detail in [5]. The reflector contains a ferrite layer and a ferroelectric layer as constituents. The key element of the device is that the material properties $\mu = \mu(H)$ and $\epsilon = \epsilon(E)$ of the composite layers are controllable in terms of the magnetic mean and the electric mean in the layers, and thus can support agile frequency attenuation. Control is implemented via local circuits which can produce rapidly changing E fields. Since the E and H fields are connected via Maxwell's equations, if the evader controls the dielectric permittivity ϵ via these local E fields, this also produces rapid changes in the magnetic permeability μ.

For our formulation we assume that the evader "controls" dielectric permittivity of the surface coatings by choosing parameters $\varepsilon = Re(\epsilon)$ from a compact admissible set $\mathcal{E} \subset \mathbb{R}_+$ in a measurable (i.e., $t \to \varepsilon(t)$ is a measurable function) time dependent manner. (Here $\mathbb{R}_+$ denotes the set of non-negative real numbers.) This produces changes in the magnetic permeability which for our initial formulation here we assume incorporates uncertainty into the reflected signal. For simplicity, we assume the real part x of the magnetic permeability $\mu = x + i\mu_i$ of the coating has uncertainty described by an Itô diffusion process X_t satisfying the stochastic differential equation

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t. \tag{2.3}$$

Here W_t denotes the standard Brownian motion and both $b = b(x)$ (the mean rate of change for $x = Re(\mu)$) and σ are non-random functions that are assumed to be Lipschitz continuous. In addition, we assume that the interrogator has control of the frequency ω of the interrogating electromagnetic signals. At each time $t \in [t_0, T]$ ($t_0 \geq 0$), the interrogator chooses parameters ω from a compact admissible set $\Omega \subset \mathbb{R}_+$.

We now can readily formulate our problem as a zero-sum differential game, where the cost functional is dependent on the expected value of the intensity of the reflected signal.

2.1 Evolution of the Expected Value of Intensity of Reflected Signal and a Dynamic Differential Game

Let $\chi(x, \varepsilon, \omega) = |R(x + i\mu_i, \varepsilon + i\epsilon_i, \omega, d)|$, where μ_i and ϵ_i denote the imaginary parts of μ and ϵ, respectively, which are assumed fixed throughout this presentation. We then define

$$\tilde{v}(t, x) = \mathbb{E}^x \left[\int_{t_0}^{t} \lambda e^{\lambda(s-t_0)} \chi(X_s, \varepsilon(s), \omega(s))\, ds + v_0(X_t) \right],$$

where $\mathbb{E}^x[\,\cdot\,]$ denotes the expectation with respect to the probability law of $\{X_t : t \geq t_0\}$ when its initial value is $X(t_0) = x$, $\lambda > 0$ is a discount parameter, and v_0 is a nonnegative function that is used to denote the initial ($t = t_0$) intensity of reflected signal.

Following a standard technique for treating integrals (see Section 10.3 of [31]), we next define the Ito diffusion Y_t in $\mathbb{R}^2$ by

$$dY_t = d\begin{pmatrix} X_t \\ Z_t \end{pmatrix} = \begin{pmatrix} b(X_t) \\ \lambda e^{\lambda(t-t_0)} \chi(X_t, \varepsilon(t), \omega(t)) \end{pmatrix} dt + \begin{pmatrix} \sigma(X_t) \\ 0 \end{pmatrix} dW_t.$$

Let $g(t, x, z) = \mathbb{E}[Z_t + v_0(X_t) \mid Y(t_0) = (x, z)^T]$, where $\mathbb{E}[\,\cdot \mid \cdot\,]$ denotes the conditional expectation. Then we have

$$\tilde{v}(t, x) = g(t, x, 0).$$

Here the generator of the Itô diffusion process $\{Y_t : t \geq t_0\}$ is

$$\mathcal{L}\phi(x, z) = b(x) \frac{\partial}{\partial x}\phi(x, z) + \frac{1}{2}\sigma^2(x) \frac{\partial^2}{\partial x^2}\phi(x, z) + \lambda e^{\lambda(t-t_0)} \chi(x, \varepsilon(t), \omega(t)) \frac{\partial}{\partial z}\phi(x, z).$$

It then follows from Section 8.1 in [31] that g satisfies the *backward Kolmogorov* equation

$$\frac{\partial}{\partial t} g = \mathcal{L} g, \quad g(t_0, x, z) = z + v_0(x). \tag{2.4}$$

A discussion of the relationship between this state and the semigroup generated by $\mathcal{L}$ can be found in [15].

Since $g = \tilde{v} + z$ is the solution to (2.4), it follows that $\tilde{v}$ satisfies

$$\begin{aligned} &\frac{\partial}{\partial t}\tilde{v}(t, x) = b(x) \frac{\partial}{\partial x}\tilde{v}(t, x) + \frac{1}{2}\sigma^2(x) \frac{\partial^2}{\partial x^2}\tilde{v}(t, x) + \lambda e^{\lambda(t-t_0)} \chi(x, \varepsilon(t), \omega(t)), \\ &\tilde{v}(t_0, x) = v_0(x). \end{aligned}$$

Now let $v(t, x) = e^{-\lambda(t-t_0)}\tilde{v}(t, x)$. It is easy to show that v satisfies

$$\begin{aligned} \frac{\partial}{\partial t} v(t, x) &= b(x) \frac{\partial}{\partial x} v(t, x) + \frac{1}{2}\sigma^2(x) \frac{\partial^2}{\partial x^2} v(t, x) - \lambda v(t, x) + \lambda \chi(x, \varepsilon(t), \omega(t)), \\ v(t_0, x) &= v_0(x). \end{aligned}$$

We note that the state v in this formulation is

$$v(t,x) = \mathbb{E}^x \left[\int_{t_0}^{t} \lambda e^{-\lambda(t-s)} \chi(X_s, \varepsilon(s), \omega(s))\, ds + e^{-\lambda(t-t_0)} v_0(X_t) \right],$$

the *expected value* of a measure of the reflected intensity.

We restrict x to be in a finite interval $[\underline{x}, \bar{x}]$, and set the boundary conditions to be zero. Thus we will consider the state equation

$$\begin{aligned} &\frac{\partial}{\partial t} v(t,x) = b(x) \frac{\partial}{\partial x} v(t,x) + \frac{1}{2}\sigma^2(x) \frac{\partial^2}{\partial x^2} v(t,x) - \lambda v(t,x) + \lambda\, \chi(x, \varepsilon(t), \omega(t)), \\ &v(t,\underline{x}) = 0,\ v(t,\bar{x}) = 0, \\ &v(t_0, x) = v_0(x). \end{aligned} \tag{2.5}$$

The objective of the game for the evader is to choose a strategy such that the intensity of the reflected signal is as small as possible while the objective for the interrogator is to choose a strategy so that the intensity of the reflected signal is as large as possible. Hence, the cost functional for a zero-sum differential game with uncertainty can be formulated by

$$\mathcal{J}(\varepsilon, \omega) = \int_{t_0}^{T} \int_{\underline{x}}^{\bar{x}} v(t, x; \varepsilon, \omega) dx dt. \tag{2.6}$$

A number of approaches have been used in the literature to study infinite-dimensional differential games. One approach is based on the theory developed by Elliott and Kalton [16] for differential games in Euclidean spaces. For example, an infinite-dimensional differential game on the infinite horizon was studied in [24] with strategies in the sense of Elliott and Kalton, and the value function of the differential game is characterized as the unique viscosity solution of the Hamilton-Jacobin-Isaacs equation. The other approach is based on the theory developed by Berkovitz [7] for differential games in Euclidean spaces, wherein the definition of strategy is a combination of "K strategies" discussed by Isaacs [22] and Friedman's lower strategy (e.g., see [19, 20]) and the definition of payoff and saddle point follows that of Krasovskii and Subbotin [25]. For example, infinite-dimensional differential games with strategies and payoff in the sense of Berkovitz were studied in [21] and [35] for finite horizon and infinite horizon, respectively, and the value function is characterized as the unique viscosity solution of Hamilton-Jacobi-Isaacs equation. It should be noted that the principle result of all of these investigations is that if the so-called Isaacs condition holds then the differential game has a value. The interested readers can refer to [34] for a readable short review on the history of differential games.

For the game that we present here, the Isaacs condition does not hold as the function $\int_{\underline{x}}^{\bar{x}} \phi(x)\chi(x, \varepsilon, \omega) dx$ are in general not quasiconvex in ε ($\in \mathcal{E}$) and quasiconcave in ω ($\in \Omega$) for any $\phi \in L^2(\underline{x}, \bar{x})$. In other words, our game may not have a value. A common approach

that is used to circumvent this difficulty is to enlarge the class of controls to include relaxed controls (e.g., see [17, 32, 41]). Hence we will consider the game in a corresponding relaxed form in the remaining of this paper.

2.2 Relaxed Differential Game

The notion of relaxed control, or generalized curve, was introduced into the calculus of variations (in the 40's) and optimal control (in the 60's) by a number of distinguished contributors such as Young [42, 43], McShane [28, 29, 30], Filippov [18] and Warga [38, 39, 40]. Since then, it has been studied by many other researchers (e.g., see [1, 10, 11, 27]).

Before we give the relaxed forms for (2.5) and (2.6), we will introduce needed theoretical background information on relaxed controls (e.g., see [17, 39, 40]). Let $C(\Omega)$ and $C(\mathcal{E})$ denote the spaces of continuous functions equipped with usual supremum norm, and $C^*(\Omega)$ and $C^*(\mathcal{E})$ be their corresponding topological dual spaces taken with the weak star topology which is equivalent to the Prohorov metric topology [9, 33] used in the static games in [2]. We define the spaces $\mathscr{P}(\Omega)$ and $\mathscr{P}(\mathcal{E})$ as the spaces of all regular probability measures defined on the Borel subsets of Ω and $\mathcal{E}$, respectively. Then with the Prohorov metric, $\mathscr{P}(\Omega)$ and $\mathscr{P}(\mathcal{E})$ are compact and convex subsets of $C^*(\Omega)$ and $C^*(\mathcal{E})$, respectively. In addition, as noted above convergence in the Prohorov metric is equivalent to weak star convergence. For more information on Prohorov metric, the interested readers can refer to [9, 33].

Let $L^1(t_0, T; C(\Omega))$ be the Banach space of Lebesgue integrable functions from $[t_0, T]$ to $C(\Omega)$ with the norm

$$\|g_\omega\|_{L^1(t_0,T;C(\Omega))} = \int_{t_0}^{T} \|g_\omega(t)\|_{C(\Omega)} dt.$$

The Banach space $L^1(t_0, T; C(\mathcal{E}))$ and its norm is similarly defined. It is known that both $L^1(t_0, T; C(\Omega))$ and $L^1(t_0, T; C(\mathcal{E}))$ are separable. We denote the topological dual of $L^1(t_0, T; C(\Omega))$ and $L^1(t_0, T; C(\mathcal{E}))$ by $L^1(t_0, T; C(\Omega))^*$ and $L^1(t_0, T; C(\mathcal{E}))^*$, respectively. By the Dunford-Pettis theorem (e.g., see [40, Theorem IV.1.8]), we have the equivalence that

$$L^1(t_0, T; C(\Omega))^* \cong L^\infty(t_0, T; C^*(\Omega))$$

and

$$L^1(t_0, T; C(\mathcal{E}))^* \cong L^\infty(t_0, T; C^*(\mathcal{E})).$$

Here $L^\infty(t_0, T; C^*(\Omega))$ is a Banach space of essentially bounded measurable functions from $[t_0, T]$ to $C^*(\Omega)$ with the norm

$$\|\Phi_\omega\|_{L^\infty(t_0,T;C^*(\Omega))} = \operatorname*{ess\,sup}_{t\in[t_0,T]} |\Phi_\omega(t)|(\Omega).$$

The Banach space $L^\infty(t_0, T; C^*(\mathcal{E})))$ and its norm is similarly defined. However, in this paper we shall consider $L^\infty(t_0, T; C^*(\Omega))$ and $L^\infty(t_0, T; C^*(\mathcal{E}))$ taken with the weak star topology.

A sequence $\{\Phi_{\omega,j}\}$ in $L^\infty(t_0,T;C^*(\Omega))$ is said to be convergent in this topology if there exists a point $\Phi_\omega \in L^\infty(t_0,T;C^*(\Omega))$ such that for any $g_\omega \in L^1(t_0,T;C(\Omega))$ we have

$$\lim_{j\to\infty}\int_{t_0}^{T}\int_\Omega g_\omega(t,\omega)\Phi_{\omega,j}(t)(d\omega)dt = \int_{t_0}^{T}\int_\Omega g_\omega(t,\omega)\Phi_\omega(t)(d\omega)dt.$$

The convergence of a sequence in $L^\infty(t_0,T;C^*(\mathcal{E}))$ with the weak star topology is similarly defined.

A relaxed control for the interrogator is a mapping $\Phi_\omega : [t_0,T] \to \mathscr{P}(\Omega)$, and this mapping is measurable (respectively, continuous) if $\int_\Omega h_\omega(\omega)\Phi_\omega(t)(d\omega)$ is measurable (respectively, continuous) function of $t \in [t_0,T]$ for every continuous real-valued function h_ω on Ω. A relaxed control for the evader $\Phi_\varepsilon : [t_0,T] \to \mathscr{P}(\mathcal{E})$ is defined similarly. We shall identify these controls which differ only on a set of measure zero. Let

$$\mathscr{R}(\Omega) = \{\Phi_\omega \mid \Phi_\omega : [t_0,T] \to \mathscr{P}(\Omega) \text{ is measurable}\}.$$

and

$$\mathscr{R}(\mathcal{E}) = \{\Phi_\varepsilon \mid \Phi_\varepsilon : [t_0,T] \to \mathscr{P}(\mathcal{E}) \text{ is measurable}\}.$$

Let $\mathscr{B}_\Omega$ and $\mathscr{B}_\mathcal{E}$ denote the unit ball of $L^\infty(t_0,T;C^*(\Omega))$ and $L^\infty(t_0,T;C^*(\mathcal{E}))$, respectively. That is,

$$\mathscr{B}_\Omega = \{\Phi \in L^\infty(t_0,T;C^*(\Omega)) \mid \|\Phi\|_{L^\infty(t_0,T;C^*(\Omega))} \le 1\}$$

and

$$\mathscr{B}_\mathcal{E} = \{\Phi \in L^\infty(t_0,T;C^*(\mathcal{E})) \mid \|\Phi\|_{L^\infty(t_0,T;C^*(\mathcal{E}))} \le 1\}.$$

Then the weak norm topology and weak star topology of $\mathscr{B}_\Omega$ (respectively, $\mathscr{B}_\mathcal{E}$) coincide, and with this topology $\mathscr{B}_\Omega$ (respectively, $\mathscr{B}_\mathcal{E}$) is a compact metric space (see [40, Theorem I.3.11 and Theorem I.3.12]). Note that for any $\Phi_\omega \in \mathscr{R}(\Omega)$ and $\Phi_\varepsilon \in \mathscr{R}(\mathcal{E})$ we have $\Phi_\omega(t)(\Omega) = 1$ and $\Phi_\varepsilon(t)(\mathcal{E}) = 1$. Hence, $\mathscr{R}(\Omega) \subset \mathscr{B}_\Omega$ and $\mathscr{R}(\mathcal{E}) \subset \mathscr{B}_\mathcal{E}$. In addition, we have the following important results.

Theorem 2.1. *(See [40, IV.2.1] or [17, Theorem 3.9]) The sets $\mathscr{R}(\Omega)$ and $\mathscr{R}(\mathcal{E})$ can be considered as closed convex subsets of the unit ball of $L^\infty(t_0,T;C^*(\Omega))$ and $L^\infty(t_0,T;C^*(\mathcal{E}))$, respectively, so with the weak star topology both $\mathscr{R}(\Omega)$ and $\mathscr{R}(\mathcal{E})$ are compact.*

Let $\Phi_\omega \in \mathscr{R}(\Omega)$ and $\Phi_\varepsilon \in \mathscr{R}(\mathcal{E})$. Then by Lemma 3.13 in [17] we know that $\Phi_\varepsilon \times \Phi_\omega$ is a measurable relaxed control on $\mathcal{E} \times \Omega$, and $\Phi_\varepsilon \times \Phi_\omega$ can be considered to belong to the unit sphere of the topological dual $L^\infty(t_0,T;C^*(\mathcal{E}\times\Omega))$ of $L^1(t_0,T;C(\mathcal{E}\times\Omega))$. With this background information on relaxed controls, we can now reformulate the state equation (2.5) in relaxed control form

$$\begin{aligned}
&\frac{\partial}{\partial t}v(t,x) = b(x)\frac{\partial}{\partial x}v(t,x) + \frac{1}{2}\sigma^2(x)\frac{\partial^2}{\partial x^2}v(t,x) - \lambda v(t,x) + f(t,x),\\
&v(t,\underline{x}) = 0,\ v(t,\bar{x}) = 0,\\
&v(t_0,x) = v_0(x),
\end{aligned} \tag{2.7}$$

where

$$f(t,x) = \lambda \int_{\Omega} \int_{\mathcal{E}} \chi(x,\varepsilon,\omega)\Phi_{\varepsilon}(t)(d\varepsilon)\ \Phi_{\omega}(t)(d\omega). \tag{2.8}$$

The cost functional corresponding to the relaxed controls Φ_ε and Φ_ω is defined by

$$\mathcal{J}(\Phi_\varepsilon, \Phi_\omega) = \int_{t_0}^{T} \int_{\underline{x}}^{\bar{x}} v(t,x;\Phi_\varepsilon,\Phi_\omega)dxdt. \tag{2.9}$$

Hence, for this relaxed formulation (2.9) with (2.7), the evader does not choose a single coating at each time t, but rather has a set of possibilities available for choices. The interrogator, in a similar approach, determines best probabilities for choices of frequency in the interrogating signals at each time t.

Remark 2.2. *From* (2.1)*, it is easy to see that* χ *is continuous on* $[\underline{x},\bar{x}] \times \mathcal{E} \times \Omega$*. By assumption both* $\mathcal{E}$ *and* Ω *are compact. Hence,* χ *is bounded. Let*

$$f_\varepsilon(t,x,\omega) = \int_{\mathcal{E}} \chi(x,\varepsilon,\omega)\Phi_\varepsilon(t)(d\varepsilon).$$

Then by the Lebesgue dominated convergence theorem we know that $f_\varepsilon(t,\cdot,\cdot)$ *is continuous on* $[\underline{x},\bar{x}]\times\Omega$ *for fixed* t*, and by the definition of relaxed controls we know* $f_\varepsilon(\cdot,x,\omega)$ *is measurable for fixed* (x,ω)*. In addition, we have*

$$\begin{aligned} |f_\varepsilon(t,x,\omega)| &\leq \|\chi\|_{C([\underline{x},\bar{x}]\times\mathcal{E}\times\Omega)}\Phi_\varepsilon(t)(\mathcal{E}) \\ &= \|\chi\|_{C([\underline{x},\bar{x}]\times\mathcal{E}\times\Omega)}. \end{aligned} \tag{2.10}$$

Thus, $f_\varepsilon \in L^\infty(t_0,T;C([\underline{x},\bar{x}]\times\Omega))$*, which implies that* $f_\varepsilon \in L^1(t_0,T;C([\underline{x},\bar{x}]\times\Omega))$*. Note that*

$$f(t,x) = \lambda \int_{\Omega} f_\varepsilon(t,x,\omega)\Phi_\omega(t)(d\omega).$$

Hence, $f(t,\cdot)$ *is continuous on* $[\underline{x},\bar{x}]$ *for fixed* t*, and* $f(\cdot,x)$ *is measurable for fixed* x*. Similarly, we find*

$$|f(t,x)| \leq \lambda\|\chi\|_{C([\underline{x},\bar{x}]\times\mathcal{E}\times\Omega)}. \tag{2.11}$$

Thus, $f \in L^\infty(t_0,T;C([\underline{x},\bar{x}]))$*. In addition, by Fubini's theorem we can exchange the order of integration in* (2.8)*.*

2.3 Existence of Saddle Points for Relaxed Differential Game

In this section we show that the relaxed form of the minmax dynamic differential game for (2.9) subject to (2.7) has a saddle point. We assume that there exists a positive constant $\sigma_{\inf}$ such that $\sigma(x) \geq \sigma_{\inf}$ for any $x \in [\underline{x},\bar{x}]$. Let $\mathbb{H} = L^2(\underline{x},\bar{x})$, $\mathbb{V} = H_0^1(\underline{x},\bar{x})$, and denote the topological dual space $\mathbb{V}^*$ by $\mathbb{V}^* = H^{-1}(\underline{x},\bar{x})$. If we identify $\mathbb{H}$ with its topological dual $\mathbb{H}^*$ then $\mathbb{V} \hookrightarrow \mathbb{H} \simeq \mathbb{H}^* \hookrightarrow \mathbb{V}^*$ forms a Gelfand triple [26]. Throughout this presentation

$\|\cdot\|_{\mathbb{H}}$ and $\|\cdot\|_{\mathbb{V}}$ and $\|\cdot\|_{\mathbb{V}^*}$ are used to denote the norms in $\mathbb{H}$, $\mathbb{V}$ and $\mathbb{V}^*$, respectively, $\langle\cdot,\cdot\rangle$ denotes the inner product in $\mathbb{H}$, and $\langle\cdot,\cdot\rangle_{\mathbb{V}^*,\mathbb{V}}$ represents the duality paring between $\mathbb{V}^*$ and $\mathbb{V}$. Following standard conventions, we use an over dot ($\dot{}$) to denote the derivative with respect to the time variable t, and use prime ($'$) to represent the derivative with respective to the space (i.e., permeability) variable x. In addition, for convenience we may use $\|\cdot\|_\infty$ to denote both the norms in $L^\infty(\underline{x},\bar{x})$ and $C([\underline{x},\bar{x}])$.

Define the sesquilinear form a on $\mathbb{V}\times\mathbb{V}$ by

$$a(\phi,\psi) = -\langle b\phi',\psi\rangle + \frac{1}{2}\langle\phi',(\sigma^2\psi)'\rangle + \lambda\langle\phi,\psi\rangle. \tag{2.12}$$

By Remark 2.2, we know that $f(t)\in C([\underline{x},\bar{x}])$. Hence, we may rewrite (3.2) in the weak form

$$\begin{aligned} &\langle\dot{v}(t),\psi\rangle_{\mathbb{V}^*,\mathbb{V}} + a(v(t),\psi) = \langle f(t),\psi\rangle, \\ &v(t_0) = v_0 \end{aligned} \tag{2.13}$$

for any $\psi\in\mathbb{V}$. Here and elsewhere $v(t)$ and $f(t)$ denote the functions $v(t,\cdot)$ and $f(t,\cdot)$, respectively.

Theorem 2.3. *Let $v_0\in\mathbb{H}$ and assume σ is Lipschitz continuous with $b\in L^\infty(\underline{x},\bar{x})$. Then there exists a unique solution v for (2.13) with $v\in H^1(t_0,T;\mathbb{V}^*)\cap L^2(t_0,T;\mathbb{V})$. In addition, there exists a positive constant κ such that for any $t\in[t_0,T]$*

$$\|v(t)\|_{\mathbb{H}}^2 \le \kappa\left(\|v_0\|_{\mathbb{H}}^2 + \int_{t_0}^t \|f(s)\|_{\mathbb{V}^*}^2 ds\right), \tag{2.14}$$

and

$$\int_{t_0}^T \|v(t)\|_{\mathbb{V}}^2 ds \le \kappa\left(\|v_0\|_{\mathbb{H}}^2 + \int_{t_0}^T \|f(s)\|_{\mathbb{V}^*}^2 ds\right). \tag{2.15}$$

Furthermore, we have $v\in C(t_0,T;\mathbb{H})$.

Proof. Note that V is continuously imbedded in H, and H is continuously imbedded in $\mathbb{V}^*$. Hence, there exists a constant $\gamma>0$ such that

$$\|\psi\|_{\mathbb{H}} \le \gamma\|\psi\|_{\mathbb{V}}, \quad \text{for any } \psi\in\mathbb{V}, \tag{2.16}$$

and

$$\|h\|_{\mathbb{V}^*} \le \gamma\|h\|_{\mathbb{H}}, \quad \text{for any } h\in\mathbb{H}. \tag{2.17}$$

Since σ is Lipshcitz continuous, $\sigma'\in L^\infty(\underline{x},\bar{x})$. Thus, by (2.16) and (2.17) we find that for any $\phi,\psi\in\mathbb{V}$ we have

$$\begin{aligned} |a(\phi,\psi)| &\le \|b\|_\infty\|\phi'\|_{\mathbb{H}}\|\psi\|_{\mathbb{H}} + \frac{1}{2}\|\sigma^2\|_\infty\|\phi'\|_{\mathbb{H}}\|\psi'\|_{\mathbb{H}} \\ &\quad + \|\sigma\|_\infty\|\sigma'\|_\infty\|\phi'\|_{\mathbb{H}}\|\psi\|_{\mathbb{H}} + \lambda\|\phi\|_{\mathbb{H}}\|\psi\|_{\mathbb{H}} \\ &\le \gamma\|b\|_\infty\|\phi\|_{\mathbb{V}}\|\psi\|_{\mathbb{V}} + \frac{1}{2}\|\sigma^2\|_\infty\|\phi\|_{\mathbb{V}}\|\psi\|_{\mathbb{V}} \\ &\quad + \gamma\|\sigma\|_\infty\|\sigma'\|_\infty\|\phi\|_{\mathbb{V}}\|\psi\|_{\mathbb{V}} + \gamma^2\lambda\|\phi\|_{\mathbb{V}}\|\psi\|_{\mathbb{V}}. \end{aligned}$$

Let $\varrho = \gamma\|b\|_\infty + \frac{1}{2}\|\sigma^2\|_\infty + \gamma\|\sigma\|_\infty\|\sigma'\|_\infty + \gamma^2\lambda$. Then by the above inequality we have

$$|a(\phi,\psi)| \le \varrho\|\phi\|_{\mathbb{V}}\|\psi\|_{\mathbb{V}}, \quad \text{for any } \phi,\psi \in \mathbb{V}. \tag{2.18}$$

For any $\psi \in \mathbb{V}$ we also obtain

$$a(\psi,\psi) \ge \left(\frac{1}{2}\sigma_{\inf}^2 - 2\theta\right)\|\psi\|_{\mathbb{V}}^2 - \frac{\|b\|_\infty^2 + \|\sigma\|_\infty^2\|\sigma'\|_\infty^2}{4\theta}\|\psi\|_{\mathbb{H}}^2.$$

Setting $\theta = \frac{1}{8}\sigma_{\inf}^2$, we have

$$a(\psi,\psi) + \alpha_{\mathbb{H}}\|\psi\|_{\mathbb{H}}^2 \ge \alpha_{\mathbb{V}}\|\psi\|_{\mathbb{V}}^2, \tag{2.19}$$

where $\alpha_{\mathbb{V}} = \frac{1}{4}\sigma_{\inf}^2$ and $\alpha_{\mathbb{H}} = \frac{\|b\|_\infty^2 + \|\sigma\|_\infty^2\|\sigma'\|_\infty^2}{4\theta}$. By Remark 2.2, we know that $f \in L^\infty(t_0,T;C([\underline{x},\bar{x}]))$. Hence, $f \in L^2(t_0,T;\mathbb{V}^*)$. Thus, by Theorem 2.1 in [4] we know that for any $v_0 \in \mathbb{H}$ there exists a unique solution v for (2.13) with $v \in H^1(t_0,T;\mathbb{V}^*) \cap L^2(t_0,T;\mathbb{V})$, and (2.14) and (2.15) hold for some positive constant κ. Furthermore, $v \in C(t_0,T;\mathbb{H})$, and thus the initial condition in (2.13) is meaningful. □

Remark 2.4. *By Remark 2.2, we know that $f(t) \in C([\underline{x},\bar{x}]) \subset \mathbb{H}$. Thus, we can easily obtain from* (2.11)

$$\|f(t)\|_{\mathbb{H}}^2 \le \kappa_{\mathbb{H}} \equiv (\bar{x} - \underline{x})\lambda^2\|\chi\|_{C([\underline{x},\bar{x}]\times\mathcal{E}\times\Omega)}^2. \tag{2.20}$$

By (2.14), (2.15), (2.17) *and* (2.20) *we find*

$$\|v(t)\|_{\mathbb{H}}^2 \le \kappa\left(\|v_0\|_{\mathbb{H}}^2 + (T-t_0)\gamma^2\kappa_{\mathbb{H}}\right), \tag{2.21}$$

and

$$\int_{t_0}^T \|v(t)\|_{\mathbb{V}}^2 ds \le \kappa\left(\|v_0\|_{\mathbb{H}}^2 + (T-t_0)\gamma^2\kappa_{\mathbb{H}}\right). \tag{2.22}$$

From (2.21) *and* (2.22), *we see that both $\|v(t)\|_{\mathbb{H}}^2$ and $\int_{t_0}^T \|v(t)\|_{\mathbb{V}}^2 ds$ are bounded by a positive constant which is independent of the choices of Φ_ω and Φ_ε.*

Remark 2.5. *Let $v(t,x)$ be the solution to* (2.13). *Then by* (2.16) *and* (2.18) *we find*

$$\begin{aligned} |\langle \dot{v}(t),\psi\rangle_{\mathbb{V}^*,\mathbb{V}}| &= |-a(v(t),\psi) + \langle f(t),\psi\rangle| \\ &\le \varrho\|v(t)\|_{\mathbb{V}}\|\psi\|_{\mathbb{V}} + \gamma\|f(t)\|_{\mathbb{H}}\|\psi\|_{\mathbb{V}}, \end{aligned}$$

which implies that

$$\begin{aligned} \|\dot{v}(t)\|_{\mathbb{V}^*} &= \sup_{\|\phi\|_{\mathbb{V}}\le 1}\{|\langle \dot{v}(t),\phi\rangle_{\mathbb{V}^*,\mathbb{V}}| \mid \phi \in \mathbb{V}\} \\ &\le \varrho\|v(t)\|_{\mathbb{V}} + \gamma\|f(t)\|_{\mathbb{H}}. \end{aligned}$$

By (2.20) *and the above equation, we obtain*

$$\|\dot{v}(t)\|_{\mathbb{V}^*}^2 \le 2\varrho^2\|v(t)\|_{\mathbb{V}}^2 + 2\gamma^2\kappa_{\mathbb{H}}.$$

Thus, by (2.22) *and integrating the above equation we have*

$$\int_{t_0}^{T} \|\dot{v}(t)\|_{\mathbb{V}^*}^2 dt \leq 2\varrho^2\kappa\|v_0\|_{\mathbb{H}}^2 + 2(T-t_0)\gamma^2\kappa_{\mathbb{H}}(\varrho^2\kappa+1). \tag{2.23}$$

From the above equation we see that $\int_{t_0}^{T} \|\dot{v}(t)\|_{\mathbb{V}^*}^2 dt$ *is bounded by a positive constant that is independent of the choices of* Φ_ω *and* Φ_ε*.*

From the definition for $\mathcal{J}$ defined in (2.9), to show $\mathcal{J}$ is separately continuous in each of its variables, it suffices to show that for given $\Phi_\varepsilon \in \mathscr{R}(\mathcal{E})$ and a sequence $\{\Phi_{\omega,j}\} \subset \mathscr{R}(\Omega)$ converging to Φ_ω in $\mathscr{R}(\Omega)$ we have

$$\lim_{j\to\infty} \int_{t_0}^{T} \int_{\underline{x}}^{\bar{x}} v(t,x;\Phi_\varepsilon,\Phi_{\omega,j})dxdt = \int_{t_0}^{T} \int_{\underline{x}}^{\bar{x}} v(t,x;\Phi_\varepsilon,\Phi_\omega)dxdt, \tag{2.24}$$

and for given $\Phi_\omega \in \mathscr{R}(\Omega)$ and a sequence $\{\Phi_{\varepsilon,j}\} \subset \mathscr{R}(\mathcal{E})$ converging to Φ_ε in $\mathscr{R}(\mathcal{E})$ we have

$$\lim_{j\to\infty} \int_{t_0}^{T} \int_{\underline{x}}^{\bar{x}} v(t,x;\Phi_{\varepsilon,j},\Phi_\omega)dxdt = \int_{t_0}^{T} \int_{\underline{x}}^{\bar{x}} v(t,x;\Phi_\varepsilon,\Phi_\omega)dxdt. \tag{2.25}$$

Actually by using (2.21), (2.22) and (2.23) and similar arguments as in [11, Lemma 2.1], we can show that (2.24) and (2.25) both hold. For convenience, we will show (2.24) in the following lemma (similar arguments can be used to establish (2.25)).

Lemma 2.6. *Let* $\Phi_\varepsilon \in \mathscr{R}(\mathcal{E})$*, and assume that the sequence* $\{\Phi_{\omega,j}\} \subset \mathscr{R}(\Omega)$ *is convergent to* Φ_ω *in* $\mathscr{R}(\Omega)$*. Then* (2.24) *holds.*

Proof. For notational convenience, we let $f_j = f(\cdot,\cdot\,;\,\Phi_\varepsilon,\Phi_{\omega,j})$ and $v_j = v(\cdot,\cdot\,;\,\Phi_\varepsilon,\Phi_{\omega,j})$. By (2.21), (2.22) and (2.23) , we know that $\{v_j\}$ is bounded in $C(t_0,T;\mathbb{H})$ and also in $L^2(t_0,T;\mathbb{V})$, and $\{\dot{v}_j\}$ is bounded in $L^2(t_0,T;\mathbb{V}^*)$. Thus, there exists a subsequence - again denoted by v_j - such that

$$v_j \to \hat{v} \text{ weakly in } L^2(t_0,T;\mathbb{V}),$$
$$\dot{v}_j \to \dot{\hat{v}} \text{ weakly in } L^2(t_0,T;\mathbb{V}^*).$$

Observe that V is also compactly imbedded in H. Hence, by Theorem 2.1 in [36] we have

$$v_j \to \hat{v} \text{ strong in } L^2(t_0,T;\mathbb{H}). \tag{2.26}$$

We further observe that $L^2(t_0,T;\mathbb{H})$ is continuously imbedded in $L^1(t_0,T;L^1(\underline{x},\bar{x}))$. Hence, by (2.26) we know that v_j is strongly convergent to $\hat{v}$ in $L^1(t_0,T;L^1(\underline{x},\bar{x}))$, which means

$$\lim_{j\to\infty} \int_{t_0}^{T} \int_{\underline{x}}^{\bar{x}} v(t,x;\Phi_\varepsilon,\Phi_{\omega,j})dxdt = \int_{t_0}^{T} \int_{\underline{x}}^{\bar{x}} \hat{v}(t,x)dxdt.$$

Thus, to complete the proof we only need to show that $\hat{v} = v(\cdot, \cdot\,;\, \Phi_\varepsilon, \Phi_\omega)$.

Let $g(t,x) = \eta(t)\psi(x)$, where $\psi \in \mathbb{V}$, and $\eta \in C^1(t_0, T)$ with $\eta(t_0) = 0$ and $\eta(T) = 0$. We set $v = v_j$ in (2.13), and then multiply (2.13) by $\eta(t)$ and integrate to find

$$\int_{t_0}^{T} \langle \dot{v}_j(t), \psi\rangle_{\mathbb{V}^*,\mathbb{V}}\, \eta(t)dt + \int_{t_0}^{T} a(v_j(t), \psi)\eta(t)dt = \int_{t_0}^{T} \langle f_j(t), \psi\rangle \eta(t)dt.$$

Integrating by parts for the first term of the above equation, we have

$$-\int_{t_0}^{T} \langle v_j(t), \psi\rangle \dot{\eta}(t)dt + \int_{t_0}^{T} a(v_j(t), \psi)\eta(t)dt = \int_{t_0}^{T} \langle f_j(t), \psi\rangle \eta(t)dt. \tag{2.27}$$

By Fubini's theorem, the right side of (2.27) can be written as

$$\begin{aligned}
&\int_{t_0}^{T} \langle f_j(t), \psi\rangle \eta(t)dt \\
&= \int_{t_0}^{T} \left[\int_{\underline{x}}^{\bar{x}} \lambda\psi(x) \left(\int_{\Omega} f_\varepsilon(t,x,\omega)\Phi_{\omega,j}(t)(d\omega) \right) dx \right] \eta(t)dt \\
&= \int_{\underline{x}}^{\bar{x}} \lambda\psi(x) \left[\int_{t_0}^{T} \int_{\Omega} \eta(t) f_\varepsilon(t,x,\omega)\Phi_{\omega,j}(t)(d\omega)dt \right] dx.
\end{aligned} \tag{2.28}$$

By Remark 2.2, we know that $f_\varepsilon \in L^\infty(t_0, T; C([\underline{x}, \bar{x}] \times \Omega))$. Since $\eta \in C^1(t_0, T)$, we have $\eta f_\varepsilon \in L^\infty(t_0, T; C([\underline{x}, \bar{x}] \times \Omega))$, which implies $\eta f_\varepsilon \in L^1(t_0, T; C([\underline{x}, \bar{x}] \times \Omega))$. Since $\Phi_{\omega,j}$ is convergent to Φ_ω in $\mathscr{R}(\Omega)$, letting $j \to \infty$, passing to the limit in (2.28) and using Fubini's theorem we find

$$\lim_{j\to\infty} \int_{t_0}^{T} \langle f_j(t), \psi\rangle \eta(t)dt = \int_{t_0}^{T} \langle f(t), \psi\rangle \eta(t)dt.$$

Now we let $j \to \infty$ and pass to the limit term by term in (2.27) to obtain

$$-\int_{t_0}^{T} \langle \hat{v}(t), \psi\rangle \dot{\eta}(t)dt + \int_{t_0}^{T} a(\hat{v}(t), \psi)\eta(t)dt = \int_{t_0}^{T} \langle f(t), \psi\rangle \eta(t)dt.$$

Integrating by parts for the first term in the above equation, we find

$$\int_{t_0}^{T} \left(\langle \dot{\hat{v}}(t), \psi\rangle_{\mathbb{V}^*,\mathbb{V}} + a(\hat{v}(t), \psi) \right) \eta(t)dt = \int_{t_0}^{T} \langle f(t), \psi\rangle \eta(t)dt. \tag{2.29}$$

Note that the class of η's for which the above holds are dense in $L^2(t_0, T)$. Hence, we have (2.29) holding for all $\eta \in L^2(t_0, T)$. Thus, we have $\hat{v}$ satisfies the first equation of (2.13). To obtain $\hat{v}(t_0) = v_0$, we may use the same arguments with arbitrary $\eta \in C^1(t_0, T)$ with $\eta(T) = 0$ but $\eta(t_0) \neq 0$. Therefore, by the uniqueness of the solution for (2.13) we have $\hat{v} = v$. □

Remark 2.7. *Since the example given in [17] shows that the identity mapping from $\mathscr{R}(\mathcal{E}) \times \mathscr{R}(\Omega) \to \mathscr{R}(\mathcal{E} \times \Omega)$ is not jointly continuous, the cost functional $\mathcal{J}$ defined by (2.9) is not jointly continuous over the space $\mathscr{R}(\mathcal{E}) \times \mathscr{R}(\Omega)$.*

Theorem 2.8. *(See [44, Corollary 3.2]) Let $\mathbb{X}$ be a nonempty compact and convex subset of a Hausdorff topological vector space, and let $\mathbb{Y}$ be a nonempty convex subset of a Hausdorff topological space, respectively. Suppose that $\mathscr{J} : \mathbb{X} \times \mathbb{Y} \to \mathbb{R}$ satisfies (i) for each fixed $x \in \mathbb{X}$, $y \longmapsto \mathscr{J}(x, y)$ is lower semicontinuous and quasiconvex; (ii) for each fixed $y \in \mathbb{Y}$, $x \longmapsto \mathscr{J}(x, y)$ is upper semicontinuouos and quasiconcave. Then we have*

$$\max_{x\in\mathbb{X}} \min_{y\in\mathbb{Y}} \mathscr{J}(x, y) = \min_{y\in\mathbb{Y}} \max_{x\in\mathbb{X}} \mathscr{J}(x, y).$$

Moreover, if $\mathbb{Y}$ is compact, then $\mathscr{J}$ has a saddle point in $\mathbb{X} \times \mathbb{Y}$.

Note that $\mathcal{J}$ of (2.9) is continuous and linear in each variable. Thus, by Theorems 2.1 and 2.8 we find that $\mathcal{J}$ has a saddle point, which is summarized in the following theorem.

Theorem 2.9. *There exists a pair of relaxed controls $\Phi^*_\omega \in \mathscr{R}(\Omega)$ and $\Phi^*_\varepsilon \in \mathscr{R}(\mathcal{E})$ such that*

$$\mathcal{J}(\Phi^*_\varepsilon, \Phi_\omega) \leq \mathcal{J}(\Phi^*_\varepsilon, \Phi^*_\omega) \leq \mathcal{J}(\Phi_\varepsilon, \Phi^*_\omega)$$

for any $\Phi_\varepsilon \in \mathscr{R}(\mathcal{E})$ and $\Phi_\omega \in \mathscr{R}(\Omega)$.

From Remark 2.7, we have that the cost functional $\mathcal{J}$ is not jointly continuous, which implies that there are challenges in carrying out standard numerical approximations (such as the delta approximation or spline approximation employed in [2] for the static case and the discretization method used in [11] for computation of relaxed optimal control) in the domain $\mathscr{R}(\mathcal{E}) \times \mathscr{R}(\Omega)$. To circumvent these difficulties, we will consider a special case of our relaxed differential game in the remainder of this paper, where we assume that both evader and interrogator have only finite number of choices at each time t. Then we develop a computational framework to obtain approximate optimal strategies for the resulting relaxed differential game.

3 Construction of Approximate Saddle Point Strategies for a Simplified Relaxed Differential Game

In this section, we assume $\Omega = \{\omega^*_1, \omega^*_2, \ldots, \omega^*_m\} \subset \mathbb{R}_+$, and $\mathcal{E} = \{\varepsilon^*_1, \varepsilon^*_2, \ldots, \varepsilon^*_l\} \subset \mathbb{R}_+$, and restrict our controls to measures of the form

$$\Phi_\omega(t) = \sum_{j=1}^{m} u_{\omega,j}(t)\Delta_{\omega^*_j}, \quad \Phi_\varepsilon(t) = \sum_{i=1}^{l} u_{\varepsilon,i}(t)\Delta_{\varepsilon^*_i}, \tag{3.1}$$

where $\Delta_{\omega^*_j}$ is the Dirac delta measure with atom at ω^*_j.

We consider the resulting relaxed differential game (which will be termed as *simplified relaxed differential game* in the remainder of this paper) with strategies and payoff in the sense of Berkovitz [7]. Specifically, in Section 3.1 we show that this game has a value and the value function is the unique viscosity solution of Hamilton-Jacobi-Isaacs equation. In Section 3.2 we employ Galerkin approximation techniques to reduce the problem into one in a finite dimensional space, and show that these approximate differential games have a value. Moreover the corresponding value functions converge pointwise to the value function of this simplified relaxed differential game. Then in Section 3.3 we use Berkovitz's method to construct optimal strategies for the approximate differential games, and show that these optimal strategies are the approximate optimal strategies for this simplified relaxed differential game.

First we introduce some necessary notation. Let $u_{\varepsilon,i} : [t_0, T] \to \mathbb{R}$ and $u_{\omega,j} : [t_0, T] \to \mathbb{R}$ be measurable nonnegative functions, $i = 1, 2, \ldots, l$ and $j = 1, 2, \ldots, m$, which satisfy $\sum_{i=1}^{l} u_{\varepsilon,i}(t) = 1$ and $\sum_{j=1}^{m} u_{\omega,j}(t) = 1$. Let

$$u_\varepsilon = (u_{\varepsilon,1}, u_{\varepsilon,2}, \ldots, u_{\varepsilon,l})^T, \text{ and } u_\omega = (u_{\omega,1}, u_{\omega,2}, \ldots, u_{\omega,l})^T.$$

Then we have

$$u_\varepsilon \in \mathcal{U}_\varepsilon[t_0, T], \quad u_\omega \in \mathcal{U}_\omega[t_0, T],$$

where for $t_0 \le s_1 < s_2 \le T$

$$\begin{aligned} \mathcal{U}_\varepsilon[s_1, s_2] &= \{u \mid u : [s_1, s_2] \to U^l \text{ is measurable}\}, \\ \mathcal{U}_\omega[s_1, s_2] &= \{u \mid u : [s_1, s_2] \to U^m \text{ is measurable}\}. \end{aligned}$$

Here for $k \in \mathbb{N}$ we define

$$U^k = \left\{ \mu = (\mu_1, \mu_2, \ldots, \mu_k)^T \in \mathbb{R}^k \ : \ \sum_{i=1}^{k} \mu_i = 1, \mu_i \ge 0 \right\}.$$

Hence, both U^l and U^m are convex and compact. For notational simplicity, we shall write $\mathcal{U}_\varepsilon$ for $\mathcal{U}_\varepsilon[t_0, T]$ and $\mathcal{U}_\omega$ for $\mathcal{U}_\omega[t_0, T]$.

The corresponding relaxed control form of the state equation (2.5) is then given by

$$\begin{aligned} &\frac{\partial}{\partial t} v(t,x) = b(x) \frac{\partial}{\partial x} v(t,x) + \frac{1}{2}\sigma^2(x) \frac{\partial^2}{\partial x^2} v(t,x) - \lambda v(t,x) + \tilde{f}(x, u_\varepsilon(t), u_\omega(t)), \\ &v(t, \underline{x}) = 0, \ v(t, \bar{x}) = 0, \\ &v(t_0, x) = v_0(x). \end{aligned} \tag{3.2}$$

Here the function $\tilde{f} : [\underline{x}, \bar{x}] \times U^l \times U^m \to \mathbb{R}$ is defined by

$$\tilde{f}(x, \mu_\varepsilon, \mu_\omega) = \lambda \sum_{j=1}^{m} \sum_{i=1}^{l} \mu_{\omega,j} \mu_{\varepsilon,i} \chi(x, \varepsilon_i^*, \omega_j^*) = \lambda \mu_\varepsilon^T B(x) \mu_\omega,$$

where $B(x)$ is a $l \times m$ matrix with its (i,j)th element being $\chi(x, \varepsilon_i^*, \omega_j^*)$. Observe that $\chi(\cdot, \varepsilon_i^*, \omega_j^*)$ is continuous in $[\underline{x}, \bar{x}]$. Hence, $\tilde{f}(\cdot, \mu_\varepsilon, \mu_\omega) \in C([\underline{x}, \bar{x}]) \subset \mathbb{H}$ and $\tilde{f}$ is bounded in the domain $[\underline{x}, \bar{x}] \times U^l \times U^m$. In the following, we denote $\tilde{f}(\cdot, \mu_\varepsilon, \mu_\omega)$ by $\mathcal{F}(\mu_\varepsilon, \mu_\omega)$ for ease in presentation. It is easily seen that $\mathcal{F} : U^l \times U^m \to \mathbb{H}$ is continuous.

3.1 Value of Differential Game

In this section, we will study our differential game with strategies and payoff in the sense of Berkovitz. Specifically we will use some of the results in [21, 35] to show that our differential game has a value and this value function is the unique viscosity solution of Hamilton-Jacobin-Isaacs equation.

Define the linear operator $\mathcal{A} : \mathbb{V} \to \mathbb{V}^*$ by $a(\phi, \psi) = \langle -\mathcal{A}\phi, \psi \rangle_{\mathbb{V}^*, \mathbb{V}}$, where a is defined in (2.12). Then (3.2) can be rewritten in the following abstract (in $\mathbb{V}^*$) form

$$\dot{v}(t) = \mathcal{A}v(t) + \mathcal{F}(u_\varepsilon(t), u_\omega(t)), \quad v(t_0) = v_0. \tag{3.3}$$

and its corresponding weak form is given by

$$\begin{aligned} &\langle \dot{v}(t), \psi \rangle_{\mathbb{V}^*, \mathbb{V}} + a(v(t), \psi) = \langle \mathcal{F}(u_\varepsilon(t), u_\omega(t)), \psi \rangle, \quad \text{for any } \psi \in \mathbb{V} \\ &v(t_0) = v_0. \end{aligned} \tag{3.4}$$

By (2.18), (2.19) and the arguments in [4, Section 2], we know that $\mathcal{A}$ generates an analytic semigroup $S(t)$ on $\mathbb{H}$, $\mathbb{V}$ and $\mathbb{V}^*$. Note that $\mathbb{V}$ is compactly embedded in $\mathbb{H}$. Hence, $S(t)$ on $\mathbb{H}$ is compact (see [14, page 394]). In addition, by Theorem 2.1 in [4], we know that for any given $v_0 \in \mathbb{H}$ there exists a unique solution v for (3.4) with $v \in H^1(t_0, T; \mathbb{V}^*) \cap L^2(t_0, T; \mathbb{V})$ given by

$$v(t) = S(t)v_0 + \int_{t_0}^{t} S(t-s)\mathcal{F}(u_\varepsilon(s), u_\omega(s))ds, \tag{3.5}$$

and there exists some positive constant κ such that

$$\|v(t)\|_{\mathbb{H}}^2 \le \kappa \left(\|v_0\|_{\mathbb{H}}^2 + \int_{t_0}^{t} \|\mathcal{F}(u_\varepsilon(s), u_\omega(s))\|_{\mathbb{V}^*}^2 ds \right), \quad t \in [t_0, T]. \tag{3.6}$$

Furthermore, we have $v \in C(t_0, T; \mathbb{H})$.

Let $\mathcal{G} : \mathbb{H} \to \mathbb{R}$ defined by $\mathcal{G}(\phi) = \displaystyle\int_{\underline{x}}^{\bar{x}} \phi(x)dx$ for any $\phi \in \mathbb{H}$. Note that $\mathbb{H}$ is continuously embedded in $L^1(\underline{x}, \bar{x})$. Hence, for any $\phi, \psi \in \mathbb{H}$ there exists a positive constant κ_g such that

$$|\mathcal{G}(\phi) - \mathcal{G}(\psi)| = \left| \int_{\underline{x}}^{\bar{x}} (\phi(x) - \psi(x))dx \right| \le \kappa_g \|\phi - \psi\|_{\mathbb{H}}.$$

Now we adjoin the differential equation

$$\dot{v}^0(t;t_0,v_0,u_\varepsilon,u_\omega)=\mathcal{G}(v(t,x;t_0,v_0,u_\varepsilon,u_\omega)),\quad v^0(t_0)=0 \tag{3.7}$$

to (3.4). Note that $\tilde{f}$ is bounded in the domain $[\underline{x},\bar{x}]\times U^l\times U^m$ and $\mathcal{F}(\mu_\varepsilon,\mu_\omega)\in\mathbb{H}$. Hence, by (3.6) we know that there exists some positive constant κ^0 (independent of the choices of u_ε and u_ω) such that

$$|v^0(t;t_0,v_0,u_\varepsilon,u_\omega)|^2\le\kappa^0(\|v_0\|^2_{\mathbb{H}}+(T-t_0)) \tag{3.8}$$

for any $t\in[t_0,T]$.

We now define the strategy for evader and interrogator in the sense of Berkovitz (e.g. see [21, 34]). A strategy Γ_ε for the evader is a choice of a sequence $\Pi_\varepsilon=\{\pi^n_\varepsilon\}$ of partitions of $[t_0,T]$ and a choice of a sequence of maps $\Gamma^{\Pi_\varepsilon}_\varepsilon=\{\Gamma^{\Pi_\varepsilon,n}_\varepsilon\}$, where $\Gamma^{\Pi_\varepsilon,n}_\varepsilon$ is described below. For notational simplicity, we will suppress the dependence on Π_ε in the notation and write Γ_ε for $\Gamma^{\Pi_\varepsilon}_\varepsilon$ and Γ^n_ε for $\Gamma^{\Pi_\varepsilon,n}_\varepsilon$. Let the partition points of π^n_ε be $t_0=t^{n_\varepsilon,0}_\varepsilon<t^{n_\varepsilon,1}_\varepsilon<\ldots<t^{n_\varepsilon,n_\varepsilon}_\varepsilon=T$ with $\|\pi^n_\varepsilon\|=\max\limits_{1\le i\le n}\{t^{n_\varepsilon,i}_\varepsilon-t^{n_\varepsilon,i-1}_\varepsilon\}\to 0$ as $n\to\infty$. Each map Γ^n_ε is a collection of maps $\{\Gamma^{n,j}_\varepsilon\}_{j=1}^{n_\varepsilon}$, where $\Gamma^{n,1}_\varepsilon\in\mathcal{U}_\varepsilon[t_0,t^{n_\varepsilon,1}_\varepsilon)$ and for $2\le j\le n_\varepsilon$,

$$\Gamma^{n,j}_\varepsilon:\mathcal{U}_\varepsilon[t_0,t^{n_\varepsilon,j-1}_\varepsilon)\times\mathcal{U}_\omega[t_0,t^{n_\varepsilon,j-1}_\varepsilon)\to\mathcal{U}_\varepsilon[t^{n_\varepsilon,j-1}_\varepsilon,t^{n_\varepsilon,j}_\varepsilon).$$

A strategy Γ_ω for the interrogator is similarly defined (by replacing the subscript ε to ω, and subscript ω to ε).

Note that a pair $(\Gamma^n_\varepsilon,\Gamma^n_\omega)$ of nth stage strategies determine uniquely a pair $(u^n_\varepsilon,u^n_\omega)\in\mathcal{U}_\varepsilon\times\mathcal{U}_\omega$ as follows. Let $\pi^n=\{t_0=t^{\hat{n},0}<t^{\hat{n},1}<\ldots<t^{\hat{n},\hat{n}}=T\}$ be the common refinement of π^n_ε and π^n_ω. The control functions $u^n_\varepsilon=(u^{n,1}_\varepsilon,u^{n,2}_\varepsilon,\ldots,u^{n,\hat{n}}_\varepsilon)$ and $u^n_\omega=(u^{n,1}_\omega,u^{n,2}_\omega,\ldots,u^{n,\hat{n}}_\omega)$, where $u^{n,j}_\varepsilon\in\mathcal{U}_\varepsilon[t^{\hat{n},j-1},t^{\hat{n},j})$ and $u^{n,j}_\omega\in\mathcal{U}_\omega[t^{\hat{n},j-1},t^{\hat{n},j})$, $j=1,2,\ldots,\hat{n}$. Let $u^n_{\varepsilon,j}$ and $u^n_{\omega,j}$ be the restriction of u^n_ε and u^n_ω to $[t_0,t^{\hat{n},j})$, respectively. On $[t^{\hat{n},0},t^{\hat{n},1})$, we set $u^{n,1}_\varepsilon=\Gamma^{n,1}_\varepsilon$ and $u^{n,1}_\omega=\Gamma^{n,1}_\omega$. Let $1\le j\le\hat{n}-1$. If there exists $1\le i\le n_\varepsilon$ such that $t^{\hat{n},j}=t^{n_\varepsilon,i}_\varepsilon$, then on $[t^{\hat{n},j},t^{\hat{n},j+1})$ we set $u^{n,j+1}_\varepsilon=\Gamma^{n,i+1}_\varepsilon(u^n_{\varepsilon,j},u^n_{\omega,j})$ and $u^{n,j+1}_\omega=\Gamma^{n,l+1}_\omega(u^n_{\varepsilon,j'},u^n_{\omega,j'})$, where l is the greatest integer such that $t^{n_\omega,l}_\omega<=t^{\hat{n},j}$ and j' is the integer such that $t^{n_\omega,l}_\omega=t^{\hat{n},j'}$. If there exists $1\le k\le n_\omega$ such that $t^{\hat{n},j}=t^{n_\omega,k}_\omega$, then on $[t^{\hat{n},j},t^{\hat{n},j+1})$ we set $u^{n,j+1}_\varepsilon=\Gamma^{n,l+1}_\varepsilon(u^n_{\varepsilon,j'},u^n_{\omega,j'})$ and $u^{n,j+1}_\omega=\Gamma^{n,k+1}_\omega(u^n_{\varepsilon,j},u^n_{\omega,j})$, where l is the greatest integer such that $t^{n_\varepsilon,l}_\varepsilon<=t^{\hat{n},j}$ and j' is the integer such that $t^{n_\varepsilon,l}_\varepsilon=t^{\hat{n},j'}$. The pair $(u^n_\varepsilon,u^n_\omega)$ determined this way is called the *nth stage outcome* of the pair $(\Gamma^n_\varepsilon,\Gamma^n_\omega)$ of nth stage strategies.

Now let $\{v_{0,n}\}$ be a sequence converging to v_0. For each n, we have the *nth stage trajectory* $\bar{v}(\cdot\,;t_0,v_{0,n},u^n_\varepsilon,u^n_\omega)=(v^0(\cdot\,;t_0,v_{0,n},u^n_\varepsilon,u^n_\omega),v(\cdot\,;t_0,v_{0,n},u^n_\varepsilon,u^n_\omega))^T$, which is the unique solution to (3.7) and (3.4) corresponding to the control functions u^n_ε and u^n_ω and initial condition $(0,v_{0,n})^T$. Note that $S(t)$ on $\mathbb{H}$ is compact. Hence, by [21, Lemma 2.2] we have the following result.

Lemma 3.1. *Let $(u^n_\varepsilon,u^n_\omega)$ be the nth stage outcome of the pair $(\Gamma^n_\varepsilon,\Gamma^n_\omega)$ of strategies on $[t_0,T]$, and $\{v_{0,n}\}$ be a sequence converging to v_0. Then the sequence $\{v(\cdot\,;t_0,v_{0,n},u^n_\varepsilon,u^n_\omega)\}$ is relatively compact in $C(t_0,T;\mathbb{H})$.*

With Lemma 3.1 and inequality (3.8), we now can define the concept of motion in the game. Any uniform limit of a subsequence of the nth stage trajectories $\{\bar{v}(\cdot\,; t_0, v_{0,n}, u_\varepsilon^n, u_\omega^n)\}_{n=1}^\infty$, where $v_{0,n} \to v_0$, and $(u_\varepsilon^n, u_\omega^n)$ is the outcome of the pair $(\Gamma_\varepsilon^n, \Gamma_\omega^n)$, is called the *motion* of the game corresponding to strategies $\Gamma_\varepsilon = \{\Gamma_\varepsilon^n\}$ and $\Gamma_\omega = \{\Gamma_\omega^n\}$ that starts from initial point (t_0, v_0). This motion is denoted by $\bar{v}[\cdot\,; t_0, v_0, \Gamma_\varepsilon, \Gamma_\omega] = (v^0[\cdot; t_0, v_0, \Gamma_\varepsilon, \Gamma_\omega], v[\cdot\,; t_0, v_0, \Gamma_\varepsilon, \Gamma_\omega])^T$. The set of all motions $\bar{v}[\,\cdot\,; t_0, v_0, \Gamma_\varepsilon, \Gamma_\omega]$ corresponding to the pair $(\Gamma_\varepsilon, \Gamma_\omega)$ is denoted by $\overline{\mathcal{V}}[\,\cdot\,; t_0, v_0, \Gamma_\varepsilon, \Gamma_\omega]$. Similarly, we use $\mathcal{V}^0[\,\cdot\,; t_0, v_0, \Gamma_\varepsilon, \Gamma_\omega]$ and $\mathcal{V}[\,\cdot\,; t_0, v_0, \Gamma_\varepsilon, \Gamma_\omega]$ denote the set of $v^0[\,\cdot\,; t_0, v_0, \Gamma_\varepsilon, \Gamma_\omega]$ and $v[\,\cdot\,; t_0, v_0, \Gamma_\varepsilon, \Gamma_\omega]$, respectively.

If the initial point of the augmented component of the trajectory is given by a nonnegative number v_0^0, then the corresponding extended trajectory with controls u_ε and u_ω is denoted by

$$\bar{v}(\,\cdot\,; t_0, \bar{v}_0, u_\varepsilon, u_\omega) = [v_0^0 + v^0(\,\cdot\,; t_0, v_0, u_\varepsilon, u_\omega), v(\,\cdot\,; t_0, v_0, u_\varepsilon, u_\omega)]^T,$$

where $\bar{v}_0 = [v_0^0, v_0]^T$.

To complete the description of the game, we need to define the payoff structure. The payoff corresponding to a pair of strategies $(\Gamma_\varepsilon, \Gamma_\omega)$ is set valued and is defined by

$$\mathcal{J}(\Gamma_\varepsilon, \Gamma_\omega; t_0, v_0) = \mathcal{V}^0[T; t_0, v_0, \Gamma_\varepsilon, \Gamma_\omega].$$

The evader tries to choose Γ_ε so as to minimize all elements of $\mathcal{J}(\Gamma_\varepsilon, \Gamma_\omega; t_0, v_0)$ and the interrogator tries to choose Γ_ω so as to maximize $\mathcal{J}(\Gamma_\varepsilon, \Gamma_\omega; t_0, v_0)$. Hence, we see that *the payoff is not required to be evaluated along a trajectory of the system.* We define

$$\mathcal{J}^-(t_0, v_0) = \sup_{\Gamma_\omega} \inf_{\Gamma_\epsilon} \mathcal{J}(\Gamma_\varepsilon, \Gamma_\omega; t_0, v_0), \quad \mathcal{J}^+(t_0, v_0) = \inf_{\Gamma_\epsilon} \sup_{\Gamma_\omega} \mathcal{J}(\Gamma_\varepsilon, \Gamma_\omega; t_0, v_0).$$

(If $\{Q_\beta\}$ is a collection of subsets of $\mathbb{R}$, then $\sup\limits_\beta Q_\beta \triangleq \sup \cup_\beta Q_\beta$ and $\inf\limits_\beta Q_\beta \triangleq \inf \cup_\beta Q_\beta$.) Then it is easy to see that $\mathcal{J}^-(t_0, v_0) \le \mathcal{J}^+(t_0, v_0)$. If $\mathcal{J}^-(t_0, v_0) = \mathcal{J}^+(t_0, v_0)$, we denote this common value by $\mathcal{J}^*(t_0, v_0)$ and say that *the game has a value equal to* $\mathcal{J}^*(t_0, v_0)$.

Let $\mathbb{Q}$ and $\tilde{\mathbb{Q}}$ be two sets of real numbers. We say that $\mathbb{Q} \ge \tilde{\mathbb{Q}}$ if for every $q \in \mathbb{Q}$ and every $\tilde{q} \in \tilde{\mathbb{Q}}$ the inequality $q \ge \tilde{q}$ holds. Also, if α is a real number and $\mathbb{Q}$ is a set, by $\alpha \ge \mathbb{Q}$ we mean that $\alpha \ge q$ for all $q \in \mathbb{Q}$. A similar meaning holds for $\alpha \le \mathbb{Q}$.

A pair of strategies $(\Gamma_\varepsilon^*, \Gamma_\omega^*)$ is said to be the *saddle point* (or *optimal strategies*) for the game with the initial point (t_0, v_0) if the inequality

$$\mathcal{J}(\Gamma_\varepsilon^*, \Gamma_\omega; t_0, v_0) \le \mathcal{J}(\Gamma_\varepsilon^*, \Gamma_\omega^*; t_0, v_0) \le \mathcal{J}(\Gamma_\varepsilon, \Gamma_\omega^*; t_0, v_0)$$

holds for all $(\Gamma_\varepsilon, \Gamma_\omega)$. Note that, if $(\Gamma_\varepsilon^*, \Gamma_\omega^*)$ is a saddle point, then $\mathcal{J}(\Gamma_\varepsilon^*, \Gamma_\omega^*; t_0, v_0)$ is a singleton and is given by

$$\mathcal{J}(\Gamma_\varepsilon^*, \Gamma_\omega^*; t_0, v_0) = \mathcal{J}^*(t_0, v_0).$$

We now consider games, trajectories and motions with varying initial points $(\tau, \varphi) \in [t_0, T] \times \mathbb{H}$. Given $\phi \in \mathbb{H}$ and $\varphi \in \mathbb{H}$, we define

$$\mathcal{H}(\mu_\varepsilon, \mu_\omega; \varphi, \phi) = \langle \phi, \mathcal{F}(\mu_\varepsilon, \mu_\omega) \rangle + \mathcal{G}(\varphi), \quad \mu_\varepsilon \in U^l, \mu_\omega \in U^m.$$

Observe that for any $\phi, \varphi \in \mathbb{H}$, $\mathcal{H}$ is continuous and linear in each variable. Since U^l and U^m are both compact and convex, we have $\mathcal{H}^-(\varphi, \phi) = \mathcal{H}^+(\varphi, \phi)$ for any $\phi, \varphi \in \mathbb{H}$, where

$$\mathcal{H}^-(\varphi, \phi) = \max_{\mu_\omega \in U^m} \min_{\mu_\varepsilon \in U^l} \mathcal{H}(\mu_\varepsilon, \mu_\omega; \varphi, \phi), \quad \mathcal{H}^+(\varphi, \phi) = \min_{\mu_\varepsilon \in U^l} \max_{\mu_\omega \in U^m} \mathcal{H}(\mu_\varepsilon, \mu_\omega; \varphi, \phi).$$

With this equality we say that *Isaacs condition* is satisfied. Let $\mathcal{H}^* = \mathcal{H}^- = \mathcal{H}^+$. Then by [21, Theorem 3.2] we have that the differential game has a value $\mathcal{J}^* = \mathcal{J}^- = \mathcal{J}^+$ and it is the unique viscosity solution of the Hamilton-Jacobi-Isaacs (HJI) equation

$$\begin{aligned} &\mathcal{J}^*_\tau + \langle \mathcal{A}\varphi, D\mathcal{J}^* \rangle + \mathcal{H}^*(\varphi, D\mathcal{J}^*) = 0, \quad (\tau, \varphi) \in (t_0, T) \times \mathbb{H} \\ &\mathcal{J}^*(T, \varphi) = 0 \end{aligned} \tag{3.9}$$

in the sense of Crandall-Lions [13]. Here $D\mathcal{J}^*$ denotes the Fréchet differential of $\mathcal{J}^*$ with respect to $\varphi \in \mathbb{H}$.

In [21, 35], Berkovitz's approach of constructing optimal strategies [7] was shown to be applicable to the infinite-dimensional differential game as well. It should be noted that the Berkovitz method involves using some feedback maps to construct the saddle point for the game, where these feedback maps are obtained by using some appropriate level sets related to the value function. Hence, to use this method one needs to compute the value function $\mathcal{J}^*$ for (3.9), which is a partial differential equation in an infinite-dimensional space with the unbounded operator $\mathcal{A}$ involved. To overcome some of these difficulties, the authors in [34] first approximate the unbounded operator by a bounded operator (e.g., the Yosida approximation), and then study the associated approximate differential games with these bounded operators and obtain their value functions. In addition, the authors showed that the value function for the original differential game is the limit of the value functions of these approximate differential games, and the saddle point for the approximate differential game is an approximate saddle point for the original differential game. Although this method makes the problem conceptually easier, one still needs to solve a partial differential equation defined in an infinite-dimensional space. In the remainder of this paper, we address this issue by employing Galerkin approximation techniques to reduce the problem to one in a finite dimensional space. This approximation technique has been used in [4] to establish a computationally feasible approximation theory for linear quadratic regulator control problems for infinite dimensional systems with unbounded input operators.

3.2 Value of Approximate Differential Game

In this section, we first employ Galerkin approximation methods to obtain a finite-dimensional approximation of the infinite dimensional system (3.4). Then we show that the associated

approximate differential games indeed have values, and these value functions converge pointwise to the original value function $\mathcal{J}^*$.

Let $\mathbb{V}^N$ be a sequence of finite-dimensional subspaces of $\mathbb{V} \subset \mathbb{H}$. We assume the following standard [4, Sec. 4] approximation condition

(H1) For any $\phi \in \mathbb{V}$, there exists a sequence ϕ^N in $\mathbb{V}^N$ such that $\|\phi^N - \phi\|_{\mathbb{V}} \to 0$ as $N \to \infty$.

Define the operator $\mathcal{A}^N : \mathbb{V}^N \to \mathbb{V}^N$ (which approximates $\mathcal{A}$) by restriction of a to $\mathbb{V}^N \times \mathbb{V}^N$; this yields

$$\langle -\mathcal{A}^N \phi, \psi \rangle = a(\phi, \psi), \quad \text{for all } \phi, \psi \in \mathbb{V}^N.$$

Let the operator $\mathcal{P}^N$ denote the usual orthogonal projection of $\mathbb{H}$ onto $\mathbb{V}^N$. That is, for $\phi \in \mathbb{H}$, we have $\mathcal{P}^N \phi \in \mathbb{V}^N$ is defined by

$$\langle \mathcal{P}^N \phi, \psi \rangle = \langle \phi, \psi \rangle, \quad \text{for all } \psi \in \mathbb{V}^N. \tag{3.10}$$

It then follows from (H1) that $\|\mathcal{P}^N \phi - \phi\|_{\mathbb{H}} \to 0$ as $N \to \infty$ for any $\phi \in \mathbb{H}$. In addition, this projection operator can readily be extended to $\mathcal{P}^N : \mathbb{V}^* \to \mathbb{V}^N$ by replacing $\langle \phi, \psi \rangle$ in (3.10) by $\langle \phi, \psi \rangle_{\mathbb{V}^*, V}$ for all $\phi \in \mathbb{V}^*$. For this family of approximations, the approximate problem corresponding to (3.3) is given by

$$\frac{dv^N(t)}{dt} = \mathcal{A}^N v^N(t) + \mathcal{F}^N(u_\varepsilon(t), u_\omega(t)), \quad v^N(t_0) = v_0^N. \tag{3.11}$$

Here $v^N(t)$ is the notation for $v^N(t, \cdot)$, $\mathcal{F}^N(u_\varepsilon(t), u_\omega(t)) = \mathcal{P}^N \mathcal{F}(u_\varepsilon(t), u_\omega(t))$, and $v_0^N = \mathcal{P}^N v_0$. Note that for any given u_ε and u_ω, $\mathcal{F}(u_\varepsilon(t), u_\omega(t)) \in C([\underline{x}, \bar{x}]) \subset \mathbb{H}$. Hence, for any $\psi \in \mathbb{V}^N$ we have $\langle \mathcal{F}^N(u_\varepsilon(t), u_\omega(t)), \psi \rangle = \langle \mathcal{F}(u_\varepsilon(t), u_\omega(t)), \psi \rangle$. Thus, the weak form of (3.11), i.e., the approximate problem corresponding to (3.4), can be formulated as finding $v^N(t) \in \mathbb{V}^N$ which satisfies

$$\begin{aligned} &\langle \frac{dv^N(t)}{dt}, \psi \rangle + a(v^N(t), \psi) = \langle \mathcal{F}(u_\varepsilon(t), u_\omega(t)), \psi \rangle, \quad \psi \in \mathbb{V}^N, \\ &v^N(t_0) = v_0^N. \end{aligned} \tag{3.12}$$

Let $\mathcal{G}^N : \mathbb{V}^N \to \mathbb{R}$ defined by $\mathcal{G}^N(\psi) = \displaystyle\int_{\underline{x}}^{\bar{x}} \psi(x) dx$ for any $\psi \in \mathbb{V}^N$. Then the approximate problem corresponding to (3.7) is given by

$$\frac{dv^{0N}(t)}{dt} = \mathcal{G}^N(v^N(t)), \quad v^{0N}(t_0) = 0. \tag{3.13}$$

Let $\{\psi_i^N\}_{i=1}^N$ be a basis of $\mathbb{V}^N$, $\Psi^N = (\psi_1^N, \psi_2^N, \ldots, \psi_N^N)^T$, and $Q^N \in \mathbb{R}^{N \times N}$ with its (i,j)th entry being $\langle \psi_j^N, \psi_i^N \rangle$. Then there exists $\nu^N(t) = (\nu_1^N(t), \nu_2^N(t), \ldots, \nu_N^N(t))^T \in \mathbb{R}^N$ such that

$$v^N(t) = (\nu^N(t))^T \Psi^N.$$

Substituting the above equality into (3.12) and letting $\psi = \psi_j^N$ for $j = 1, 2, \ldots, N$, we can obtain the matrix representation A^N of operator $\mathcal{A}^N$ (that is, $\mathcal{A}^N v^N = (A^N \nu^N)^T \Psi^N$) and the vector representation $F^N(u_\varepsilon(t), u_\omega(t))$ of $\mathcal{F}^N(u_\varepsilon(t), u_\omega(t))$ (that is, $\mathcal{F}^N(u_\varepsilon(t), u_\omega(t)) = (F^N(u_\varepsilon(t), u_\omega(t)))^T \Psi^N$) with respect to the basis $\{\psi_i\}_{i=1}^N$. These are given by

$$A^N = (Q^N)^{-1} \mathscr{A}^N, \text{ where } \mathscr{A}^N \in \mathbb{R}^{N\times N} \text{ with its } (i,j) \text{ element being } a(\psi_j^N, \psi_i^N),$$

and

$$F^N(u_\varepsilon(t), u_\omega(t)) = (Q^N)^{-1}(\langle \mathcal{F}(u_\varepsilon(t), u_\omega(t)), \psi_1^N\rangle, \ldots, \langle \mathcal{F}(u_\varepsilon(t), u_\omega(t)), \psi_N^N\rangle)^T,$$

respectively. Let ν_0^N be the vector representation of v_0^N with respect to the basis $\{\psi_i\}_{i=1}^N$. Then we obtain

$$\nu_0^N = (Q^N)^{-1}(\langle v_0, \psi_1^N\rangle, \langle v_0, \psi_2^N\rangle, \ldots, \langle v_0, \psi_N^N\rangle)^T.$$

We can reformulate (3.11) in terms of the system of ordinary differential equations

$$\frac{d\nu^N(t)}{dt} = A^N \nu^N(t) + F^N(u_\varepsilon(t), u_\omega(t)), \quad \nu^N(t_0) = \nu_0^N, \tag{3.14}$$

which has a unique solution. Hence, (3.11) also has a unique solution, which can be written as

$$v^N(t) = e^{t\mathcal{A}^N} \mathcal{P}^N v_0 + \int_{t_0}^t e^{(t-s)\mathcal{A}^N} \mathcal{F}^N(u_\varepsilon(s), u_\omega(s)) ds. \tag{3.15}$$

Let $G^N : \mathbb{R}^N \to \mathbb{R}$ defined by $G^N(\eta) = \sum_{j=1}^N \left(\int_{\underline{x}}^{\overline{x}} \psi_j^N(x) dx \right) \eta_j$ for any $\eta = (\eta_1, \eta_2, \ldots, \eta_N)^T \in \mathbb{R}^N$. Then (3.13) can be rewritten as

$$\frac{d\nu^{0N}(t)}{dt} = G^N(\nu^N(t)), \quad \nu^{0N}(t_0) = 0. \tag{3.16}$$

That is, we set $\nu^{0N} \equiv v^{0N}$.

The motion of corresponding Nth approximate differential game (in $\mathbb{V}^N$) associated with a pair of strategies $(\Gamma_\varepsilon, \Gamma_\omega)$ is denoted by

$$\bar{v}^N[\,\cdot\,; t_0, v_0^N, \Gamma_\varepsilon, \Gamma_\omega] = (v^{0N}[\,\cdot\,; t_0, v_0^N, \Gamma_\varepsilon, \Gamma_\omega], v^N[\,\cdot\,; t_0, v_0^N, \Gamma_\varepsilon, \Gamma_\omega])^T$$

and the set of all $\bar{v}^N[\,\cdot\,; t_0, v_0^N, \Gamma_\varepsilon, \Gamma_\omega]$ is represented by $\overline{\mathcal{V}}^N[\,\cdot\,; t_0, v_0^N, \Gamma_\varepsilon, \Gamma_\omega]$. Similarly, $\mathcal{V}^N[\,\cdot\,; t_0, v_0^N, \Gamma_\varepsilon, \Gamma_\omega]$ and $\mathcal{V}^{0N}[\,\cdot\,; t_0, v_0^N, \Gamma_\varepsilon, \Gamma_\omega]$ denote the set of $v^N[\,\cdot\,; t_0, v_0^N, \Gamma_\varepsilon, \Gamma_\omega]$ and $v^{0N}[\,\cdot\,; t_0, v_0^N, \Gamma_\varepsilon, \Gamma_\omega]$, respectively. The approximate payoff corresponding to a pair of strategies $(\Gamma_\varepsilon, \Gamma_\omega)$ is given by

$$\mathcal{J}^N(\Gamma_\varepsilon, \Gamma_\omega; t_0, v_0^N) = \mathcal{V}^{0N}[T; t_0, v_0^N, \Gamma_\varepsilon, \Gamma_\omega]. \tag{3.17}$$

Instead of directly consider the Nth approximate differential game, we may also consider its corresponding matrix representation (i.e., the differential game in $\mathbb{R}^N$). The motion for the

corresponding matrix representation of the Nth approximate game associated with a pair of strategies $(\Gamma_\varepsilon, \Gamma_\omega)$ is denoted by

$$\bar{\nu}^N[\,\cdot\,; t_0, \nu_0^N, \Gamma_\varepsilon, \Gamma_\omega] = (\nu^{0N}[\,\cdot\,; t_0, \nu_0^N, \Gamma_\varepsilon, \Gamma_\omega], \nu^N[\,\cdot\,; t_0, \nu_0^N, \Gamma_\varepsilon, \Gamma_\omega])^T$$

and the set of all $\bar{\nu}^N[\,\cdot\,; t_0, v_0^N, \Gamma_\varepsilon, \Gamma_\omega]$ is represented by $\overline{V}^N[\,\cdot\,; t_0, v_0^N, \Gamma_\varepsilon, \Gamma_\omega]$. Similarly, $V^N[\,\cdot\,; t_0, v_0^N, \Gamma_\varepsilon, \Gamma_\omega]$ and $V^{0N}[\,\cdot\,; t_0, v_0^N, \Gamma_\varepsilon, \Gamma_\omega]$ denote the set of $v^N[\,\cdot\,; t_0, v_0^N, \Gamma_\varepsilon, \Gamma_\omega]$ and $v^{0N}[\,\cdot\,; t_0, v_0^N, \Gamma_\varepsilon, \Gamma_\omega]$, respectively. Then we have

$$v^N[\,\cdot\,; t_0, v_0^N, \Gamma_\varepsilon, \Gamma_\omega] = (\nu^N[\,\cdot\,; t_0, \nu_0^N, \Gamma_\varepsilon, \Gamma_\omega])^T \Psi^N.$$

Thus, we obtain

$$v^{0N}[\,\cdot\,; t_0, v_0^N, \Gamma_\varepsilon, \Gamma_\omega] = \nu^{0N}[\,\cdot\,; t_0, \nu_0^N, \Gamma_\varepsilon, \Gamma_\omega]. \tag{3.18}$$

Therefore, the payoff (3.17) can be equivalently written as

$$J^N(\Gamma_\varepsilon, \Gamma_\omega; t_0, \nu_0^N) = V^{0N}[T; t_0, \nu_0^N, \Gamma_\varepsilon, \Gamma_\omega].$$

That is, for any given $v_0^N = (\nu_0^N)^T \Psi^N$ we have

$$\mathcal{J}^N(\Gamma_\varepsilon, \Gamma_\omega; t_0, v_0^N) = J^N(\Gamma_\varepsilon, \Gamma_\omega; t_0, \nu_0^N) \tag{3.19}$$

holds for any pair of strategies $(\Gamma_\varepsilon, \Gamma_\omega)$.

We now consider the approximate differential game (in $\mathbb{R}^N$) with varying initial points $(\tau, \xi) \in [t_0, T] \times \mathbb{R}^N$. we define

$$J^{N-}(\tau, \xi) = \sup_{\Gamma_\omega} \inf_{\Gamma_\epsilon} J^N(\Gamma_\varepsilon, \Gamma_\omega; \tau, \xi), \quad J^{N+}(\tau, \xi) = \inf_{\Gamma_\epsilon} \sup_{\Gamma_\omega} J^N(\Gamma_\varepsilon, \Gamma_\omega; \tau, \xi).$$

Then J^{N-} and J^{N+} are uniformly Lipschitz continuous on bounded subsets of $[t_0, T) \times \mathbb{R}^N$ (see [7, Theorem 7.2]). For any given $\xi, \eta \in \mathbb{R}^N$, we define

$$H^N(\mu_\varepsilon, \mu_\omega; \xi, \eta) = \eta^T F^N(\mu_\varepsilon, \mu_\omega) + G^N(\xi), \quad \mu_\varepsilon \in U^l, \mu_\omega \in U^m.$$

Observe that for any given $\eta, \xi \in \mathbb{R}^N$, H^N is continuous and linear in each variable. Since U^l and U^m are both compact and convex, we have $H^{N-}(\xi, \eta) = H^{N+}(\xi, \eta)$ for any $\xi, \eta \in \mathbb{R}^N$, where

$$H^{N-}(\xi, \eta) = \max_{\mu_\omega \in U^m} \min_{\mu_\varepsilon \in U^l} H^N(\mu_\varepsilon, \mu_\omega; \xi, \eta), \quad H^{N+}(\xi, \eta) = \min_{\mu_\varepsilon \in U^l} \max_{\mu_\omega \in U^m} H^N(\mu_\varepsilon, \mu_\omega; \xi, \eta).$$

Let $H^{N*} = H^{N-} = H^{N+}$. Then we know that there exists a value $J^{N*} = J^{N-} = J^{N+}$ and it is the unique viscosity solution of the HJI equation given by (see [8])

$$\begin{aligned} &J_\tau^{N*} + (A^N \xi)^T \nabla J^{N*} + H^{N*}\left(\xi, \nabla J^{N*}\right) = 0, \quad (\tau, \xi) \in (t_0, T) \times \mathbb{R}^N, \\ &J^{N*}(T, \xi) = 0. \end{aligned} \tag{3.20}$$

Thus, the Nth approximate differential game (in $\mathbb{V}^N$) with initial point $(\tau, \psi) \in [t_0, T] \times \mathbb{V}^N$ also has a value and it is given by $\mathcal{J}^{N*}(\tau, \psi) = J^{N*}(\tau, \xi)$, where ξ is the vector representation of ψ with respect to the basis $\{\psi_i^N\}_{i=1}^N$ (i.e., $\psi = \xi^T \Psi^N$).

In the conclusion of this section, we show that the value function for the Nth approximate differential game converges pointwise to the value function of our simplified relaxed differential game. We first recall the following convergence result, which is standard in the literature (e.g., see [4, Lemma 4.1] or [26, Chapter III]).

Lemma 3.2. *Let u_ε and u_ω be given. Suppose (H1) is satisfied and $v_0 \in \mathbb{H}$. If $v^N(t; t_0, \mathcal{P}^N v_0, u_\varepsilon, u_\omega) \in \mathbb{V}^N$, $t \geq t_0$, satisfies* (3.12)*, then we have*

$$\|v^N(t; t_0, \mathcal{P}^N v_0, u_\varepsilon, u_\omega) - v(t; t_0, v_0, u_\varepsilon, u_\omega)\|_{\mathbb{H}} \to 0 \text{ as } N \to \infty$$

and

$$\int_{t_0}^{t} \|v^N(s; t_0, \mathcal{P}^N v_0, u_\varepsilon, u_\omega) - v(s; t_0, v_0, u_\varepsilon, u_\omega)\|_{\mathbb{V}}^2 ds \to 0 \text{ as } N \to \infty$$

uniformly in $t \in [t_0, T]$.

In addition, by using Trotter-Kato theorem we have the following important convergence result (see [4, Lemma 4.3]).

Lemma 3.3. *For all $\phi \in \mathbb{H}$, we have*

$$\|e^{t\mathcal{A}^N} \mathcal{P}^N \phi - S(t)\phi\|_{\mathbb{H}} \to 0 \text{ as } N \to \infty, \tag{3.21}$$

where the convergence is uniform on bounded t-interval.

Using Lemma 3.3 and the same arguments as those in [34, Lemma 3.2] we can obtain a stronger result than that presented in Lemma 3.3.

Lemma 3.4. *For any relatively compact set $\mathbb{H}_c \subset \mathbb{H}$,*

$$\sup_{\phi \in \mathbb{H}_c} \sup_{t_0 \leq t \leq T} \|e^{t\mathcal{A}^N} \mathcal{P}^N \phi - S(t)\phi\|_{\mathbb{H}} \to 0 \text{ as } N \to \infty. \tag{3.22}$$

Lemma 3.5. *Suppose (H1) is satisfied. Then for every $v_0 \in \mathbb{H}$, we have*

$$\sup_{(u_\varepsilon, u_\omega) \in \mathcal{U}_\varepsilon \times \mathcal{U}_\omega} \sup_{t_0 \leq t \leq T} \|v^N(t; t_0, \mathcal{P}^N v_0, u_\varepsilon, u_\omega) - v(t; t_0, v_0, u_\varepsilon, u_\omega)\|_{\mathbb{H}} \to 0 \tag{3.23}$$

and

$$\sup_{(u_\varepsilon, u_\omega) \in \mathcal{U}_\varepsilon \times \mathcal{U}_\omega} \sup_{t_0 \leq t \leq T} |v^{0N}(t; t_0, \mathcal{P}^N v_0, u_\varepsilon, u_\omega) - v^0(t; t_0, v_0, u_\varepsilon, u_\omega)| \to 0. \tag{3.24}$$

as $N \to \infty$.

Proof. Let $\mathcal{F}(U^l \times U^m) = \{\mathcal{F}(\mu_\varepsilon, \mu_\omega) \mid (\mu_\varepsilon, \mu_\omega) \in U^l \times U^m\}$. Note that $\mathcal{F} : U^l \times U^m \to \mathbb{H}$ is continuous and $U^l \times U^m$ is compact. Hence, $\mathcal{F}(U^l \times U^m)$ is compact. By (3.5) and (3.15) we have

$$\begin{aligned}
&\|v^N(t; t_0, \mathcal{P}^N v_0, u_\varepsilon, u_\omega) - v(t; t_0, v_0, u_\varepsilon, u_\omega)\|_{\mathbb{H}} \\
&\quad \le \|e^{t\mathcal{A}^N}\mathcal{P}^N v_0 - S(t)v_0\|_{\mathbb{H}} \\
&\qquad + \left\| \int_{t_0}^{t} \left[e^{(t-s)\mathcal{A}^N}\mathcal{P}^N \mathcal{F}(u_\varepsilon(s), u_\omega(s)) - S(t-s)\mathcal{F}(u_\varepsilon(s), u_\omega(s)) \right] ds \right\|_{\mathbb{H}}.
\end{aligned}$$

Thus, by the above inequalities, the compactness of $\mathcal{F}(U^l \times U^m)$ and Lemma 3.4 we obtain the desired result (3.23).

Note that $\mathbb{H}$ is continuously embedded in $L^1(\underline{x}, \bar{x})$. Hence, there exists a positive constant α such that

$$\begin{aligned}
&|v^{0N}(t; t_0, \mathcal{P}^N v_0, u_\varepsilon, u_\omega) - v^0(t; t_0, v_0, u_\varepsilon, u_\omega)| \\
&\quad = \left| \int_{\underline{x}}^{\bar{x}} \left(v^N(t, x; t_0, \mathcal{P}^N v_0, u_\varepsilon, u_\omega) - v(t, x; t_0, v_0, u_\varepsilon, u_\omega) \right) dx \right| \\
&\quad \le \alpha \|v^N(t; t_0, \mathcal{P}^N v_0, u_\varepsilon, u_\omega) - v(t; t_0, v_0, u_\varepsilon, u_\omega)\|_{\mathbb{H}}.
\end{aligned}$$

Thus, by (3.23) and the above inequality we have the desired result (3.24). □

Theorem 3.6. *For every $\tau \in [t_0, T]$ and $\varphi \in \mathbb{H}$, we have $\mathcal{J}^{N*}(\tau, \mathcal{P}^N\varphi) \to \mathcal{J}^*(\tau, \varphi)$.*

Proof. Fixed $(\tau, \varphi) \in [t_0, T] \times \mathbb{H}$. For any given $\delta > 0$, there exist a strategy Γ_ε^0 and a strategy Γ_ω^0 such that

$$\sup_{\Gamma_\omega} \mathcal{J}(\Gamma_\varepsilon^0, \Gamma_\omega; \tau, \varphi) < \mathcal{J}^*(\tau, \varphi) + \frac{\delta}{4} \tag{3.25}$$

and

$$\inf_{\Gamma_\varepsilon} \mathcal{J}(\Gamma_\varepsilon, \Gamma_\omega^0; \tau, \varphi) > \mathcal{J}^*(\tau, \varphi) - \frac{\delta}{4}. \tag{3.26}$$

Then for any given positive integer k, there exists a strategy Γ_ω^k and an associated motion $\bar{v}^k[\,\cdot\,; \tau, \mathcal{P}^k\varphi, \Gamma_\varepsilon^0, \Gamma_\omega^k]$ such that

$$\mathcal{J}^{k*}(\tau, \mathcal{P}^k\varphi) \le \sup_{\Gamma_\omega} \mathcal{J}^k(\Gamma_\varepsilon^0, \Gamma_\omega; \tau, \mathcal{P}^k\varphi) < v^{0k}[T; \tau, \mathcal{P}^k\varphi, \Gamma_\varepsilon^0, \Gamma_\omega^k] + \frac{\delta}{4}, \tag{3.27}$$

and there exists a strategy Γ_ε^k and an associated motion $\bar{v}^k[\,\cdot\,; \tau, \mathcal{P}^k\varphi, \Gamma_\varepsilon^k, \Gamma_\omega^0]$ such that

$$\mathcal{J}^{k*}(\tau, \mathcal{P}^k\varphi) \ge \inf_{\Gamma_\varepsilon} \mathcal{J}^k(\Gamma_\varepsilon, \Gamma_\omega^0; \tau, \mathcal{P}^k\varphi) > v^{0k}[T; \tau, \mathcal{P}^k\varphi, \Gamma_\varepsilon^k, \Gamma_\omega^0] - \frac{\delta}{4}. \tag{3.28}$$

By Lemma 3.5 we know that there exists a positive integer N_0 such that for any $N \geq N_0$ we have

$$\sup_{(u_\varepsilon,u_\omega)\in\mathcal{U}_\varepsilon\times\mathcal{U}_\omega}\ \sup_{\tau\leq t\leq T}|v^{0N}(t;\tau,\mathcal{P}^N\varphi,u_\varepsilon,u_\omega)-v^0(t;\tau,\varphi,u_\varepsilon,u_\omega)|\leq\frac{\delta}{6}.$$

For any given $N \geq N_0$, let $\bar{v}^N[\,\cdot\,;\tau,\mathcal{P}^N\varphi,\Gamma_\varepsilon^0,\Gamma_\omega^N]$ be the uniform limit of subsequence $\{\bar{v}^N(\,\cdot\,;\tau,\mathcal{P}^N\varphi,u_\varepsilon^{0,n_N},u_\omega^{N,n_N})\}$ of sequence $\{\bar{v}^N(\,\cdot\,;\tau,\mathcal{P}^N\varphi,u_\varepsilon^{0,n},u_\omega^{k,n})\}$. By Lemma 3.1 and inequality (3.8), we know that there exists a subsequence $\left\{\left(u_\varepsilon^{0,n_{N_N}},u_\omega^{N,n_{N_N}}\right)\right\}$ of sequence $\{(u_\varepsilon^{0,n_N},u_\omega^{N,n_N})\}$ such that $\left\{\bar{v}\left(\,\cdot\,;\tau,\varphi,u_\varepsilon^{0,n_{N_N}},u_\omega^{N,n_{N_N}}\right)\right\}$ converges uniformly, and we denote this uniform limit by $\bar{v}[\,\cdot\,;\tau,\varphi,\Gamma_\varepsilon^0,\Gamma_\omega^N]$. Then for any given $N \geq N_0$ we can always find sufficiently large n_{N_N} such that

$$\begin{aligned}
&\left|v^{0N}[T;\tau,\mathcal{P}^N\varphi,\Gamma_\varepsilon^0,\Gamma_\omega^N]-v^0[T;\tau,\varphi,\Gamma_\varepsilon^0,\Gamma_\omega^N]\right|\\
&\quad\leq\left|v^{0N}[T;\tau,\mathcal{P}^N\varphi,\Gamma_\varepsilon^0,\Gamma_\omega^N]-v^{0N}(T;\tau,\mathcal{P}^N\varphi,u_\varepsilon^{0,n_{N_N}},u_\omega^{N,n_{N_N}})\right|\\
&\qquad+\left|v^{0N}(T;\tau,\mathcal{P}^N\varphi,u_\varepsilon^{0,n_{N_N}},u_\omega^{N,n_{N_N}})-v^0(T;\tau,\varphi,u_\varepsilon^{0,n_{N_N}},u_\omega^{N,n_{N_N}})\right|\\
&\qquad+\left|v^0(T;\tau,\varphi,u_\varepsilon^{0,n_{N_N}},u_\omega^{N,n_{N_N}})-v^0[T;\tau,\varphi,\Gamma_\varepsilon^0,\Gamma_\omega^N]\right|\\
&\quad\leq\frac{\delta}{2}.
\end{aligned}\tag{3.29}$$

Similarly, for N sufficiently large we can also find a motion $\bar{v}[\,\cdot\,;\tau,\varphi,\Gamma_\varepsilon^N,\Gamma_\omega^0]$ such that

$$\left|v^{0N}[T;\tau,\mathcal{P}^N\varphi,\Gamma_\varepsilon^N,\Gamma_\omega^0]-v^0[T;\tau,\varphi,\Gamma_\varepsilon^N,\Gamma_\omega^0]\right|\leq\frac{\delta}{2}.\tag{3.30}$$

By (3.25), (3.27) and (3.29) we find that for N sufficiently large we have

$$\begin{aligned}
&\mathcal{J}^{N*}(\tau,\mathcal{P}^N\varphi)-\mathcal{J}^*(\tau,\varphi)\\
&\quad<v^{0N}[T;\tau,\mathcal{P}^N\varphi,\Gamma_\varepsilon^0,\Gamma_\omega^N]+\frac{\delta}{4}-\left(\sup_{\Gamma_\omega}\mathcal{J}(\Gamma_\varepsilon^0,\Gamma_\omega;\tau,\varphi)-\frac{\delta}{4}\right)\\
&\quad\leq v^{0N}[T;\tau,\mathcal{P}^N\varphi,\Gamma_\varepsilon^0,\Gamma_\omega^N]-v^0[T;\tau,\varphi,\Gamma_\varepsilon^0,\Gamma_\omega^N]+\frac{\delta}{2}\\
&\quad\leq\delta.
\end{aligned}\tag{3.31}$$

Similarly, by (3.26), (3.28) and (3.30) we find that for sufficiently large N we obtain

$$\begin{aligned}
&\mathcal{J}^*(\tau,\varphi)-\mathcal{J}^{N*}(\tau,\mathcal{P}^N\varphi)\\
&\quad<\inf_{\Gamma_\varepsilon}\mathcal{J}(\Gamma_\varepsilon,\Gamma_\omega^0;\tau,\varphi)+\frac{\delta}{4}-\left(v^{0N}[T;\tau,\mathcal{P}^N\varphi,\Gamma_\varepsilon^N,\Gamma_\omega^0]-\frac{\delta}{4}\right)\\
&\quad\leq v^0[T;\tau,\varphi,\Gamma_\varepsilon^N,\Gamma_\omega^0]-v^{0N}[T;\tau,\mathcal{P}^N\varphi,\Gamma_\varepsilon^N,\Gamma_\omega^0]+\frac{\delta}{2}\\
&\quad\leq\delta.
\end{aligned}\tag{3.32}$$

Thus, the desired convergence result follows from (3.31) and (3.32). □

3.3 Saddle Point for the Approximate Differential Game

In the previous section, we have shown that when N is sufficiently large the value function $\mathcal{J}^{N*}$ for the Nth approximate differential game is an approximation of value function $\mathcal{J}^*$ for our simplified relaxed differential game. Hence, instead of seeking optimal strategies for our simplified relaxed differential game, we find optimal strategies for its associated approximate differential game. First we define the approximate optimal strategies for our simplified relaxed differential game (see the corresponding definition in [34]).

Definition 3.7. *A strategy Γ_ε^* is said to be δ-optimal (for the evader) of the simplified relaxed differential game with initial point (t_0, v_0) if*

$$\sup_{\Gamma_\omega} \mathcal{J}(\Gamma_\varepsilon^*, \Gamma_\omega; t_0, v_0) \leq \mathcal{J}^*(t_0, v_0) + \delta.$$

A strategy Γ_ω^* is said to be δ-optimal (for the interrogator) of the simplified relaxed differential game with initial point (t_0, v_0) if

$$\inf_{\Gamma_\varepsilon} \mathcal{J}(\Gamma_\varepsilon, \Gamma_\omega^*; t_0, v_0) \geq \mathcal{J}^*(t_0, v_0) - \delta.$$

If Γ_ε^* and Γ_ω^* are δ-optimal, then we say that $(\Gamma_\varepsilon^*, \Gamma_\omega^*)$ constitute a δ-saddle point (or δ-optimal strategies) for the simplified relaxed differential game with initial point (t_0, v_0).

The below result shows that the optimal strategies for the Nth approximate differential game are indeed approximate optimal strategies for the simplified relaxed differential game when N is sufficiently large.

Theorem 3.8. *Let $(\Gamma_\varepsilon^{N*}, \Gamma_\omega^{N*})$ be a saddle point for the Nth approximate differential game with initial point $(t_0, \mathcal{P}^N v_0)$. Then for every $\delta > 0$ there exists N_δ such that for all $N \geq N_\delta$, Γ_ε^{N*} and Γ_ω^{N*} are δ-optimal strategies for the evader and interrogator, respectively, for the simplified relaxed differential game with initial point (t_0, v_0).*

Proof. Since $(\Gamma_\varepsilon^{N*}, \Gamma_\omega^{N*})$ is a saddle point for the Nth approximate differential game with initial point $\big(t_0, \mathcal{P}^N v_0\big)$, we have

$$\sup_{\Gamma_\omega} \mathcal{J}^N(\Gamma_\varepsilon^{N*}, \Gamma_\omega; t_0, \mathcal{P}^N v_0) \leq \mathcal{J}^{N*}(t_0, \mathcal{P}^N v_0) \leq \inf_{\Gamma_\varepsilon} \mathcal{J}^N(\Gamma_\varepsilon, \Gamma_\omega^{N*}; t_0, \mathcal{P}^N v_0). \tag{3.33}$$

For any given $\delta > 0$, there exists a strategy Γ_ω^0 and a motion $\bar{v}[\,\cdot\,; t_0, v_0, \Gamma_\varepsilon^{N*}, \Gamma_\omega^0]$ such that

$$\sup_{\Gamma_\omega} \mathcal{J}(\Gamma_\varepsilon^{N*}, \Gamma_\omega; t_0, v_0) < v^0[T; t_0, v_0, \Gamma_\varepsilon^{N*}, \Gamma_\omega^0] + \frac{\delta}{4}, \tag{3.34}$$

and there exists a strategy Γ_ε^0 and a motion $\bar{v}[\,\cdot\,; t_0, v_0, \Gamma_\varepsilon^0, \Gamma_\omega^{N*}]$ such that

$$\inf_{\Gamma_\varepsilon} \mathcal{J}(\Gamma_\varepsilon, \Gamma_\omega^{N*}; t_0, v_0) > v^0[T; t_0, v_0, \Gamma_\varepsilon^0, \Gamma_\omega^{N*}] - \frac{\delta}{4}. \tag{3.35}$$

By Theorem 3.6, we know that for any given $\delta > 0$ there exists a positive integer $N_{\delta 1}$ such that for any $N \geq N_{\delta 1}$ we have

$$\mathcal{J}^*(t_0, v_0) - \frac{\delta}{2} < \mathcal{J}^{N*}(t_0, \mathcal{P}^N v_0) < \mathcal{J}^*(t_0, v_0) + \frac{\delta}{2}. \tag{3.36}$$

In addition, by using the same arguments as those in Theorem 3.6 we know that there exists a positive integer $N_{\delta 2}$ such that for any given $N \geq N_{\delta 2}$ we can always find a corresponding motion $\bar{v}^N[\,\cdot\,; t_0, \mathcal{P}v_0, \Gamma_\varepsilon^{N*}, \Gamma_\omega^0]$ so that

$$\left| v^{N0}[T; t_0, \mathcal{P}v_0, \Gamma_\varepsilon^{N*}, \Gamma_\omega^0] - v^0[T; t_0, v_0, \Gamma_\varepsilon^{N*}, \Gamma_\omega^0] \right| < \frac{\delta}{4}, \tag{3.37}$$

and a corresponding motion $\bar{v}^N[\,\cdot\,; t_0, v_0, \Gamma_\varepsilon^0, \Gamma_\omega^{N*}]$ so that

$$\left| v^{N0}[T; t_0, v_0, \Gamma_\varepsilon^0, \Gamma_\omega^{N*}] - v^0[T; t_0, v_0, \Gamma_\varepsilon^0, \Gamma_\omega^{N*}] \right| < \frac{\delta}{4}. \tag{3.38}$$

Let $N_\delta = \max\{N_{\delta 1}, N_{\delta 2}\}$. Then by (3.33), (3.34), (3.36) and (3.37) we find

$$\sup_{\Gamma_\omega} \mathcal{J}(\Gamma_\varepsilon^{N*}, \Gamma_\omega; t_0, v_0) < \mathcal{J}^*(t_0, v_0) + \delta, \tag{3.39}$$

and by (3.33), (3.35), (3.36) and (3.38) we obtain

$$\inf_{\Gamma_\varepsilon} \mathcal{J}(\Gamma_\varepsilon, \Gamma_\omega^{N*}; t_0, v_0) > \mathcal{J}^*(t_0, v_0) - \delta. \tag{3.40}$$

Thus, for all $N \geq N_\delta$, Γ_ε^{N*} and Γ_ω^{N*} are δ-optimal strategies for the evader and interrogator, respectively, for the simplified relaxed differential game with initial point (t_0, v_0). □

By Theorem 3.8 we know that in order to construct approximate optimal strategies for our simplified relaxed differential game, it is sufficient to construct a saddle point for the Nth approximate differential game. Let $\bar{\xi} = (\xi^0, \xi)^T, \bar{\eta} = (\eta^0, \eta)^T \in \mathbb{R}^{N+1}$, where $\xi^0, \eta^0 \in \mathbb{R}$ and $\xi, \eta \in \mathbb{R}^N$. Then for any $\mu_\varepsilon \in U^l$ and $\mu_\omega \in U^m$ we define

$$\overline{H}^N(\mu_\varepsilon, \mu_\omega; \bar{\xi}, \bar{\eta}) = \eta^T F^N(\mu_\varepsilon, \mu_\omega) + \eta^0 G^N(\xi). \tag{3.41}$$

Observe that for any given $\bar{\xi}, \bar{\eta} \in \mathbb{R}^{N+1}$, $\overline{H}^N$ is continuous and linear in each variable. Since U^l and U^m are both compact and convex, we have

$$\max_{\mu_\omega \in U^m} \min_{\mu_\varepsilon \in U^l} \overline{H}^N(\mu_\varepsilon, \mu_\omega; \bar{\xi}, \bar{\eta}) = \min_{\mu_\varepsilon \in U^l} \max_{\mu_\omega \in U^m} \overline{H}^N(\mu_\varepsilon, \mu_\omega; \bar{\xi}, \bar{\eta}),$$

which is essential to construct the optimal strategies.

In the remainder of this section we will follow the Berkovitz method [7] to construct the saddle point strategies for the Nth approximate differential game. For notational simplicity,

we suppress the dependence of the strategies on N. Let (t_0, ν_0^N) be the initial point of the game and $\alpha^N = J^{N*}(t_0, \nu_0^N)$. We define the level sets

$$\mathbb{C}^0(\alpha^N) = \{(t, \bar{\xi}) \in [t_0, T] \times \mathbb{R}^{N+1} \mid \xi^0 + J^{N*}(t, \xi) \geq \alpha^N\}$$
$$\mathbb{C}_0(\alpha^N) = \{(t, \bar{\xi}) \in [t_0, T] \times \mathbb{R}^{N+1} \mid \xi^0 + J^{N*}(t, \xi) \leq \alpha^N\}.$$

Since J^{N*} is continuous and $(t_0, (0, \nu_0^N)^T) \in \mathbb{C}^0(\alpha^N) \cap \mathbb{C}_0(\alpha^N)$, the level sets $\mathbb{C}_0(\alpha^N)$ and $\mathbb{C}^0(\alpha^N)$ are nonempty closed sets.

First we define a feedback strategy. Let $p_\varepsilon = \{p_\varepsilon^n\}$, where $p_\varepsilon^n : [t_0, T] \times \mathbb{R}^{N+1} \to U^l$ is called a *feedback map* for the evader. The function p_ε determines a strategy $\Gamma_\varepsilon = \Gamma_\varepsilon(p_\varepsilon)$ in the game with initial point (t_0, ν_0^N) as follows. Let $\pi^n = \{t_0 = t^{n,0} < t^{n,1} < \ldots < t^{n,n} = T\}$ be the partition of $[t_0, T]$ which divides $[t_0, T]$ into n subintervals of same length. Define $\Gamma_\varepsilon^{n,1} = p_\varepsilon^n(t_0, (0, \nu_0^N)^T)$. For $2 \leq j \leq n$ and $(u_\varepsilon, u_\omega) \in \mathcal{U}_\varepsilon[t_0, t^{n,j-1}) \times \mathcal{U}_\omega[t_0, t^{n,j-1})$, we define

$$\Gamma_\varepsilon^{n,j}(u_\varepsilon, u_\omega) = p_\varepsilon^n(t^{n,j-1}, \bar{\nu}^N(t^{n,j-1})),$$

where $\bar{\nu}^N(t) = (\nu^{0N}(t; t_0, \nu_0^N, u_\varepsilon, u_\omega), \nu^N(t; t_0, \nu_0^N, u_\varepsilon, u_\omega))^T$ (i.e., $\nu^{0N}(t; t_0, \nu_0^N, u_\varepsilon, u_\omega)$ is the solution to (3.16), and $\nu^N(t; t_0, \nu_0^N, u_\varepsilon, u_\omega)$ is the solution to (3.14)) for $t_0 \leq t \leq t^{n,j-1}$. The strategy $\Gamma_\varepsilon = \Gamma_\varepsilon(p_\varepsilon)$ determined this way is called a *feedback strategy* for the evader corresponding to feedback map p_ε. In a similar fashion, for every sequence of feedback maps $p_\omega = \{p_\omega^n\}$ for the interrogator with $p_\omega^n : [t_0, T] \times \mathbb{R}^{N+1} \to U^m$ we can define an associated feedback strategy $\Gamma_\omega = \Gamma_\omega(p_\omega)$ for the interrogator.

We next turn to construct, for a given partition π^n, optimal feedback maps, which are extremal to the level sets. If $(t, \bar{\xi}) \in \mathbb{C}_0(\alpha^N)$, then we define $p_\varepsilon^{n*}(t, \bar{\xi}) = \mu_{\varepsilon,0}$, which is an arbitrarily fixed vector in U^l. Otherwise, consider the set $\mathbb{C}_{0t} = \{\bar{\zeta} \mid (t, \bar{\zeta}) \in \mathbb{C}_0(\alpha^N)\}$ (which is not empty due to [7, Lemma 8.3]) and choose a point $\bar{\zeta}^* \in \mathbb{C}_{0t}$ such that $\|\bar{\zeta}^* - \bar{\xi}\| = \min_{\eta \in \mathbb{C}_{0t}} \|\bar{\zeta} - \bar{\xi}\|$. Let $\bar{\eta}^* = \bar{\xi} - \bar{\zeta}^*$. Define $p_\varepsilon^{n*}(t, \bar{\xi}) = \mu_\varepsilon^*$, where $(\mu_\varepsilon^*, \mu_\omega^*)$ is any saddle point of the local game with payoff $\overline{H}^N(\mu_\varepsilon, \mu_\omega; \bar{\xi}, \bar{\eta}^*)$ defined by (3.41). The feedback strategy corresponding to this sequence of feedback maps $p_\varepsilon^* = \{p_\varepsilon^{n*}\}$ is denoted by Γ_ε^*. In an analogous way, using $\mathbb{C}^0(\alpha^N)$ in place of $\mathbb{C}_0(\alpha^N)$, one can construct the sequence of feedback maps $\{p_\omega^{n*}\}$ and the corresponding feedback strategy Γ_ω^* for the interrogator. Then by [7, Lemma 10.1] we know that all motions $\bar{\nu}^N[t; t_0, \nu_0^N, \Gamma_\varepsilon^*, \Gamma_\omega]$ and $\bar{\nu}^N[t; t_0, \nu_0^N, \Gamma_\varepsilon, \Gamma_\omega^*]$ lie entirely in $\mathbb{C}_0(\alpha^N)$ and $\mathbb{C}^0(\alpha^N)$, respectively. That is,

$$\nu^{0N}[t; t_0, \nu_0^N, \Gamma_\varepsilon^*, \Gamma_\omega] + J^{N*}(t, \nu^N[t; t_0, \nu_0^N, \Gamma_\varepsilon^*, \Gamma_\omega]) \leq J^{N*}(t_0, \nu_0^N)$$

and

$$\nu^{0N}[t; t_0, \nu_0^N, \Gamma_\varepsilon, \Gamma_\omega^*] + J^{N*}(t, \nu^N[t; t_0, \nu_0^N, \Gamma_\varepsilon, \Gamma_\omega^*]) \geq J^{N*}(t_0, \nu_0^N).$$

Note that $J^{N*}(T, \nu^N[T; t_0, \nu_0^N, \Gamma_\varepsilon^*, \Gamma_\omega]) = 0$ and $J^{N*}(T, \nu^N[T; t_0, \nu_0^N, \Gamma_\varepsilon, \Gamma_\omega^*]) = 0$. Hence, we have

$$\nu^{0N}[T; t_0, \nu_0^N, \Gamma_\varepsilon^*, \Gamma_\omega] \leq J^{N*}(t_0, \nu_0^N) \leq \nu^{0N}[T; t_0, \nu_0^N, \Gamma_\varepsilon, \Gamma_\omega^*],$$

By (3.18) and (3.19), we can equivalently write the above inequalities as

$$v^{0N}[T; t_0, v_0^N, \Gamma_\varepsilon^*, \Gamma_\omega] \leq \mathcal{J}^{N*}(t_0, v_0^N) \leq v^{0N}[T; t_0, v_0^N, \Gamma_\varepsilon, \Gamma_\omega^*].$$

Thus, the pair $(\Gamma_\varepsilon^*, \Gamma_\omega^*)$ constitutes a saddle point for the Nth approximate differential game with initial point (t_0, v_0^N) .

4 Concluding Remarks

In this paper a two-player zero-sum dynamic differential game is considered in the context of electromagnetic pursuit-evasion games. The formulations are a natural extension of the corresponding static games developed in [2] in the context of relaxed strategies. The cost functional here is based on the expected value of the intensity of the reflected signal. We established that the resulting relaxed differential game has a saddle point, which is found to be difficult to compute due to lack of the joint continuity of the cost functional on the relaxed controls. To overcome this difficulty, we then consider a special case of this game in which both players have only finite number of control choices available at each time. This simplified game is studied with strategies and payoff in the sense of Berkovitz [7], and is shown to have a value where the value function is the unique viscosity solution of Hamilton-Jacobi-Isaacs equation in the sense of Crandall-Lions [13]. We then employ Galerkin approximation techniques to reduce this simplified game to ones in finite dimensional spaces. The value functions of the associated approximate differential games are shown to converge pointwise to the value function of this simplified game. In addition, we show the optimal strategies for the approximate differential game are approximate optimal strategies for the simplified game. It is useful to observe that the computational framework presented in this paper is also applicable to differential games with state governed by a general semilinear evolution equations such as those studied in [21, 34].

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